Continuous-time quantum walks (CTQW) over finite group schemes is investigated, where it is shown that some properties of a CTQW over a group scheme defined on a finite group $G$ induces a CTQW over group scheme defined on $G/H$, where $H$ is a normal subgroup of $G$ with prime index. This reduction can be helpful in analyzing CTQW on underlying graphs of group schemes. Even though this claim is proved for normal subgroups with prime index (using the Clifford’s theorem from representation theory), it is checked in some examples that for other normal subgroups or even non-normal subgroups, the result is also true! It means that CTQW over the graph on $G$, starting from any arbitrary vertex, is isomorphic to the CTQW over the quotient graph on $G/H$ if we take the sum of the amplitudes corresponding to the vertices belonging to the same cosets.

Keywords: Continuous-time quantum walks (CTQW); quotient graphs; normal subgroup.

1. Introduction

Quantum walks have recently been introduced and investigated with the hope that they may be useful in constructing new efficient quantum algorithms (for reviews of quantum walks, see Refs. 1–3). A study of random walks on simple graph is well-known in physics (see Ref. 4). Recent studies of quantum walks on more general graphs were described in Refs. 1, 5–17. Some of these works study the problem in the important context of algorithmic problems on graphs and suggest that quantum
walks is a promising algorithmic technique for designing future quantum algorithms. In particular, the study of continuous-time quantum walks (CTQW) on graphs has shown promising applications in the algorithmic and implementation aspects. Quantum walks have applications other than in the design of new algorithms. In Refs. 18–22, CTQWs on graphs have been used to efficiently transfer quantum states with perfect or optimal fidelity.

In this paper, we consider CTQW over more general graphs that are underlying graphs of group association schemes. In fact, the theory of association schemes has its origin in the design of statistical experiments. The algebraic formulation of association schemes was done by Bose and Mesner who introduced an algebra generated by the adjacency matrices of the association scheme, known as Bose-Mesner (BM) algebra. We will employ the algebraic structures of these graphs in order to calculate the corresponding probability amplitudes, in terms of the parameters of the corresponding association scheme such as the diameter of the scheme and the so-called first eigenvalue matrix $P$ (which is related to the character table of the corresponding group in the case of group association schemes). As we will see, the preference of this employment is that we are able to give analytical formulas for probability amplitudes of CTQW over these graphs in terms of the irreducible characters of the corresponding group. Then, we use the representation theory to explain aspects of CTQW on certain classes of graphs called underlying graphs of association schemes. We employ the theorem of Clifford from representation theory to show that a CTQW over a group association scheme defined on a finite group $G$ induces a CTQW over group association scheme defined on $G/H$, where $H$ is a normal subgroup of $G$ with prime index. In fact, it is shown that if we sum the probability amplitudes corresponding to the vertices belonging to the same cosets of $H$ in $G$, then the resulted amplitudes are the same as the amplitudes of CTQW over the underlying graph of the quotient group association scheme defined on $G/H$, up to a computable time scale. This reduction can be helpful in analyzing CTQW on underlying graphs of group association schemes. The structure of the paper is as follows: In Sec. 2, some preliminary facts about association schemes, especially group association schemes and their algebraic formulation are reviewed. In Sec. 3, CTQW over underlying graphs of group association schemes is investigated where it is shown that CTQW on finite group $G$ induces a CTQW on $G/H$ where $H$ is a normal subgroup of $G$ with prime index $p$, if we sum the probability amplitudes corresponding to the vertices belonging to the same cosets of $H$ in $G$. Section 4 is devoted to CTQW over some examples of underlying graphs of group association schemes and the corresponding coset $G/H$ graphs. The paper ends with a brief conclusion and an Appendix.

2. Preliminaries

In this section, we recall some preliminary facts about association schemes, especially group association schemes and their algebraic properties. The reader is referred to Ref. 23 for more details.
2.1. Association schemes

Let $V$ and $E$ be vertex and edge sets of a regular graph, respectively. An association scheme with $d$ associate classes on the finite set $V$ is a set of matrices $A_0, A_1, \ldots, A_d$ such that

$$A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k,$$

(1)

$A_0 = I$, and the sum of $A_i$ is the all-one matrix $J$. From (1), it is seen that the adjacency matrices $A_0, A_1, \ldots, A_d$ form a basis for a commutative algebra $A$ known as the Bose-Mesner algebra of the association scheme. This algebra has a second basis $E_0, \ldots, E_d$ (known as primitive idempotents) so that

$$E_0 = \frac{1}{N} J, \quad E_i E_j = \delta_{ij} E_i, \quad \sum_{i=0}^{d} E_i = I.$$

(2)

Let $P$ and $Q$ be the matrices relating the two bases for $A$:

$$A_i = \sum_{j=0}^{d} P_{ij} E_j, \quad 0 \leq i \leq d,$$

$$E_i = \frac{1}{N} \sum_{j=0}^{d} Q_{ij} A_j, \quad 0 \leq i \leq d.$$

(3)

Then, clearly we have

$$A_i E_j = P_{ij} E_j,$$

$$PQ = QP = NI.$$

(4)

which shows that the $P_{ij}$ is the $j$th eigenvalue of $A_i$ and that the columns of $E_j$ are the corresponding eigenvectors. Thus, $m_i = \text{rank}(E_i)$ is the multiplicity of the eigenvalue $P_{ij}$ of $A_i$ (provided that $P_{ij} \neq P_{kj}$ for $k \neq i$).

2.1.1. Group schemes

Group schemes are particular association schemes for which the vertex set contains elements of a finite group $G$ and the $i$th adjacency matrix $A_i$ is defined as:

$$A_i = \mathcal{C}_i := \sum_{g \in C_i} g,$$

where $C_0 = \{e\}, C_1, \ldots, C_d$ are the conjugacy classes of $G$ and $g$ is considered in the regular representation of the group. The corresponding idempotents $E_0, \ldots, E_d$ are the projection operators as

$$E_k = \frac{\chi_k(1)}{|G|} \sum_{\alpha \in G} \chi_k(\alpha^{-1}) \alpha$$

(5)
where $\chi_k$ is the $k$th irreducible character of $G$. Thus eigenvalues of adjacency matrices $A_k$ and idempotents $E_k$ are given by

$$P_{ik} = \frac{\kappa_i \chi_k(\alpha_i)}{d_k}, \quad Q_{ik} = d_i \chi_i(\alpha_k)$$

(6)

respectively, where $d_j = \chi_j(1)$ is the dimension of the irreducible character $\chi_j$ and $\kappa_k = |C_k|$ is the $k$th valency of the graph.

3. Continuous-Time Quantum Walk on Group Schemes

CTQW on a graph $\Gamma$ is defined by replacing master equation of continuous-time classical random walk (Kolmogorov’s equation) with Schrödinger’s equation\textsuperscript{25} where, $L$ (the Laplacian of the graph) is chosen as the Hamiltonian of the walk. Let $W$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by vertex set $V$ of $\Gamma$, and whose entries are in $\mathbb{C}$ (i.e. $W = \mathbb{C}^n$, with $n = |V|$). With each $\alpha \in V$ we associate a ket defined by $|\alpha\rangle$, then $\{|\alpha\rangle : \alpha \in V\}$ becomes a complete orthonormal basis of $W$. Let $|\phi(t)\rangle$ be a time-dependent state of the quantum process on graph. The wave evolution of the quantum walk is given by

$$i \hbar \frac{d}{dt} |\phi(t)\rangle = \mathcal{H} |\phi(t)\rangle$$

(7)

where we assume $\hbar = 1$, and $|\phi_0\rangle$ is the initial amplitude wave function of the particle. The solution is given by $|\phi_0(t)\rangle = e^{-i\mathcal{H}t}|\phi_0\rangle$. It is more natural to deal with the Laplacian of the graph, defined as $L = A - D$, where $D$ is a diagonal matrix with entries $D_{jj} = \text{deg}(\alpha_j)$. This is because we can view $L$ as the generator matrix that describes an exponential distribution of waiting times at each vertex. But on $d$-regular graphs, $D = (1/d)I$, and since $A$ and $D$ commute, we get

$$e^{-i\mathcal{H}t} = e^{-iL/2} = e^{it/d}e^{-iAt}.$$  (8)

This introduces an irrelevant phase factor in the wave evolution. Hence we can consider $\mathcal{H} = A = A_1$. Then we have

$$|\phi_0(t)\rangle = e^{-iAt}|\phi_0\rangle = e^{-i\sum_{i=0}^{d-1} P_i E_i t} |\phi_0\rangle$$

(9)

where using the algebra of idempotents (2–7) amplitude of wave function can be written as

$$|\phi_0(t)\rangle = \sum_{i=0}^{d} e^{-iP_i t} E_i |\phi_0\rangle.$$  (10)

Finally we get the following expression for the amplitude of observing the particle at vertex $\beta \in C_k$ at time $t$

$$\langle \beta | \phi_0(t) \rangle = \sum_{i=0}^{d} e^{-iP_i t} \langle \beta | E_i |\phi_0\rangle = \frac{1}{n} \sum_{i=0}^{d} e^{-iP_i t} Q_{ki}.$$  (11)
Now, we discuss CTQW on group schemes with real and complex representations separately. In a finite group $G$ with real conjugacy classes $C_0 = \{e\}, C_1, \ldots, C_d$ in the sense that $C(\alpha) = C(\alpha^{-1})$ for all $\alpha \in G$, all of the irreducible characters $\chi_i$ are real. Then, by using (6) and (11), for every state $\beta \in C_k$ we have

$$\langle \beta | \phi_0(t) \rangle = \frac{1}{|G|} \sum_{i=0}^{d} d_i e^{-i\frac{t}{\kappa_1} \chi_i(\alpha_k)}, \quad \alpha_k \in C_k,$$

(12)

where $\kappa_1$ is the valency of the graph. In the cases that all of the conjugacy classes are not real and hence some of the irreducible representations are complex, we encounter with directed underlying graphs or non-symmetric association schemes. In these cases, one can generate a symmetric association scheme out of a given non-symmetric association scheme (see the Appendix A of Ref. 13) so that the amplitude of observing the CTQW at vertex $\beta \in C_k$ at time $t$ is given by

$$\langle \beta | \phi_0(t) \rangle = \begin{cases} 
\frac{1}{|G|} \sum_{i=0}^{d} d_i e^{-i\frac{t}{\kappa_1} \chi_i(\alpha_k)} & \text{for real representations,} \\
\frac{1}{|G|} \sum_{i=0}^{d} d_i e^{-i\frac{t}{\kappa_1} \chi_i(\alpha_k)} & \text{for complex representations.}
\end{cases}$$

(13)

for more details, see Ref. 13.

The following lemma shows that some properties of a CTQW on finite group can be deduced from a CTQW on $G/H$ where $H$ is a normal subgroup of $G$ with prime index $p$, i.e. $|G/H| = p$. This reduction can be helpful in analyzing CTQW on groups.

**Lemma.** Let $H \leq G$ be a normal subgroup of prime index $p$ and $T = (t_0 = e, t_1, \ldots, t_{p-1})$ a transversal of $G/H$ (it is well known that any finite group of prime order is cyclic, so we have $G/H \cong Z_p$ and $t_l = t^l$ for $l = 0, 1, \ldots, p-1$) so that $G = H \cup t_1 H \cup t_2 H \cup \cdots t_{p-1} H$. Then, for each $\beta \in t_l H$, we have the following correspondence between amplitudes of CTQW over group graph $G$ and those of the smaller quotient graph $G/H$ as follows

$$\sum_{h \in H} \langle t_l h | \phi_0(t) \rangle = \left\langle t_l | \phi_0 \left( \frac{\kappa_1}{\kappa_1} t \right) \right\rangle, \quad l = 0, 1, \ldots, p - 1,$$

(14)

where, $|\phi_0(t)\rangle = e^{-iAt}|\phi_0\rangle$ with $|\phi_0\rangle$ as initial state of CTQW over $G$, whereas $|\phi_0'(t)\rangle = e^{-iAt'}|\phi_0'\rangle$ with $|\phi_0'\rangle$ as initial state of CTQW over $G/H$ ($A'$ is the adjacency matrix of the quotient graph $G/H$); $\kappa_1 \equiv |\tilde{C}_1|$ is the valency of the underlying graph of the group scheme $G/H$.

**Proof.** By using the result (12), we can write

$$\sum_{h \in H} \langle t_l h | \phi_0(t) \rangle = \frac{1}{|G|} \sum_{i=0}^{d} d_i e^{-i\frac{t}{\kappa_1} \chi_i(\alpha)} \sum_{h \in H} \chi_i(th).$$

(15)
In order to evaluate the sum $\sum_{h \in H} \chi(tlh)$ we use the theorem of Clifford\cite{21} in its most general form which deals with the restriction of an irreducible representation of a group $G$ to a normal subgroup $H \triangleleft G$. Therefore, we recall this theorem in the Appendix A. In the $\sum_{h \in H} \chi(tlh)$, $\chi$ is scalar or non-scalar; for scalar representations $\chi_{\text{scalar}}$, one can show that

$$\sum_{h \in H} \chi_{\text{scalar}}(t_l h) = \begin{cases} |H| \chi_{\text{scalar}}(t), & \text{if } (\chi_{\text{scalar}} \downarrow H) \text{ be trivial}, \\ 0, & \text{otherwise}, \end{cases}$$

where $\chi_{\text{scalar}} \downarrow H$ denotes the restriction of $\chi$ to the subgroup $H$. To see this, we define $I(g)$ as

$$I(g) := \sum_{h \in H} \chi(gh).$$

Then, we have

$$I(g') = \sum_{h \in H} \chi(g'h) = \sum_{h \in H} \chi(gh'h') = \sum_{h \in H} \chi(gh) \chi(h') = I(g) \chi(h').$$

On the other hand, we have

$$I(g') = \sum_{h \in H} \chi(g'h) = \sum_{h' \in H} \chi(gh') = I(g).$$

Therefore, we obtain

$$I(g) (\chi(h') - 1) = 0.$$

Then, for scalar representations $\chi_{\text{scalar}}$ which are non-trivial on the subgroup $H$, we have $I(g) = \sum_{h \in H} \chi_{\text{scalar}}(gh) = 0$.

For non-scalar representations $\chi_{\text{non-scalar}}$, one can write

$$\sum_{h \in H} \chi_{\text{non-scalar}}(t_l h) = \sum_{h \in H} (\psi \uparrow G)(t_l h),$$

where $\psi$ is the character of an irreducible representation $\rho$ of $H$ and $\psi \uparrow G$ denotes the induction of $\psi$ to the group $G$ (see for example Ref. 11). If $\rho \cong \rho^i$ for $i = 0, 1, \ldots, p - 1$, where $\rho^i(g) := \rho(t_l^{-1} g t_i)$. Then, the Clifford’s theorem guarantees that

$$(\rho \uparrow G)^A = \bigoplus_{i=0}^{p-1} \lambda_i \cdot \tilde{\rho} \rightarrow (\psi \uparrow G)(t_l h) = \tilde{\rho} \sum_{i=0}^{p-1} \omega^i = 0,$$

where, $\tilde{\rho}$ is one of the $p$ pairwise inequivalent extensions of $\rho$ to $G$ and $\omega := e^{-2\pi i/p}$. In the above relation, we have used the fact that $\rho \uparrow G$ and its similarity transformation $(\rho \uparrow G)^A = A(\rho \uparrow G)A^{-1}$ have the same character $\psi \uparrow G$.  

If $\rho \neq \rho^i$, we use the fact that any induced representation $\rho \uparrow G$ is block off diagonal for elements belonging to the cosets $t_l H$, with $t_l \neq e$ ($e$ is the identity
element of the group $G$), i.e. we have
\[
(\rho \uparrow G)(t_i h) = \begin{pmatrix}
0 & * & \cdots & * \\
* & 0 & \cdots & * \\
\cdots & \cdots & \cdots & \cdots \\
* & * & \cdots & 0
\end{pmatrix},
\]
for all $h \in H$ and $t_i \neq e$, whereas we have
\[
(\rho \uparrow G)(h) = \begin{pmatrix}
* & 0 & \cdots & 0 \\
0 & * & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & *
\end{pmatrix},
\]
for all $h \in H$. This implies that, the induced characters $\psi \uparrow G$, are zero on cosets $t_i H$ with $t_i \neq e$. Then, we can write
\[
\sum_{h \in H} (\psi \uparrow G)(t_i h) = 0, \quad \text{if } t_i \neq e.
\]
In order to evaluate the sum $\sum_{h \in H}(\psi \uparrow G)(h)$, we use the second part of the Clifford’s theorem to write
\[
(\psi \uparrow G) \downarrow H = \bigoplus_{i=0}^{p-1}\psi^{t_i} = p\psi \rightarrow \sum_{h \in H}(\psi \uparrow G)(h) = p \sum_{h \in H}(\psi)(h) = 0,
\]
where we have used the fact that $\psi^{t_i}(h) = \psi(t_i h t_i^{-1}) = \psi(h)$ for every $h \in H$. Also, we have used the equality $\sum_{h \in H}(\psi)(h) = 0$ which can be easily deduced as follows: By defining
\[
I := \sum_{h \in H}(\rho)(h),
\]
we can write
\[
\rho(h')I = \sum_{h \in H}\rho(h')\rho(h) = \sum_{h \in H}\rho(h'h) = I.
\]
Then, we have
\[
(\rho(h') - 1)I = 0.
\]
Since we have $\rho^{t_i} \neq \rho$, then $\rho(h') \neq 1$ for all $h' \in H$, i.e. there is a $h' \in H$ so that $\rho(h') \neq 1$. Then, the above equality implies that $I = \sum_{h \in H}\rho(h) = 0$ and so we have $\sum_{h \in H}(\psi)(h) = 0$.

From the above arguments, we deduce that the sum in Eq. (15) can be rewritten as
\[
\sum_{h \in H}\langle t_i h | \phi_0(t) \rangle = \frac{|H|}{|G|} \sum_{\chi} d_{\chi} e^{i\frac{k_1}{k_1}\chi(t)} \chi(t_i) = \langle t_i | \phi_0 (\kappa_1 t) \rangle,
\]
where the sum is taken over all scalar representations \( \chi \) of \( G \) which are trivial on \( H \), i.e. all the scalar representations of \( G \) with kernel \( H \). Clearly these representations are irreducible representations of \( G / H \cong \mathbb{Z}_p \). Then, by comparing the right hand side of (16) with Eq. (12), we conclude that CTQW over \( G \) induces CTQW over \( G / H \) up to a time scale \( \kappa / \kappa \), if we take the sum of the amplitudes of CTQW over \( G \) corresponding to the elements belonging to the same cosets.

Even though we have proved the above result for the case that the normal subgroup \( H \) has prime index \( p \), as it will be seen in the examples given in the next section, for other normal or non-normal subgroups, the same result is also true!

### 3.1. Examples

In this section, as examples, we consider CTQW over underlying graphs of cyclic group, dihedral group and quaternion group.

**A. Cyclic group**

If \( \mathbb{Z}_n \) is cyclic group we have \( \mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\} \). In the following we consider only the case of even \( n = 2k \), the case of odd \( n \) can be considered similarly. The character table is given by

\[
\begin{array}{cccccc}
 g & 0 & 1 & \cdots & n-1 \\
 \hline
 \chi_0 & 1 & 1 & \cdots & 1 \\
 \chi_1 & 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\
 \vdots & \vdots & \vdots & \cdots & \vdots \\
 \chi_{n/2} & 1 & -1 & 1 & \cdots & (-1)^{n-1} = -1 \\
 \vdots & \vdots & \vdots & \cdots & \vdots \\
 \chi_{n-1} & 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} \\
\end{array}
\]

By choosing \( \mathbb{Z}_{n/2} = \{0, 2, 4, \ldots, n-2\} \) as a subgroup of \( \mathbb{Z}_n \), and using: Eq. (12), one can evaluate the sum of the amplitudes of CTQW corresponding to the elements belonging to the same cosets as follows:

\[
\sum_{j=0}^{k-1} \langle 2j | \phi_0(t) \rangle = \frac{1}{n} \left\{ e^{-2i\chi_0(1)t} \sum_{j=0}^{k-1} \chi_0(2j) + e^{-2i\chi_{n/2}(1)t} \sum_{j=0}^{k-1} \chi_{n/2}(2j) \right. \\
+ \left. \sum_{l \neq 0, n/2} e^{-2i(\chi_l(1)+\bar{\chi}_l(1))t} \sum_{j=0}^{k-1} (\chi_l(2j) + \bar{\chi}_l(2j)) \right\}
\]
\[
\frac{1}{n} \left\{ \frac{n}{2} e^{-2it} + \frac{n}{2} e^{2it} + \sum_{l \neq 0, n/2}^{n-1} e^{-2it} \frac{2\pi}{C_0} \sum_{j=0}^{k-1} (\omega^{2lj} + \omega^{-2lj}) \right\} = \frac{1}{2} (e^{-2it} + e^{2it}) = \cos(2t). 
\]

Similarly, one can easily show that
\[
\sum_{j=0}^{k-1} \langle 2j + 1 | \phi_0(t) \rangle = -i \sin(2t).
\]

As it was expected from the Clifford’s theorem (see the lemma on page 8), only the representations \( \chi_0 \) and \( \chi_{n/2} \) which are trivial on \( H = Z_{n/2} \) remain in the above sums so that the obtained results are the amplitudes of CTQW over \( G/H \cong Z_2 \). The above result indicates that the sum of the amplitudes of CTQW (associated with \( Z_n \)) over the cosets of \( Z_{n/2} \) in \( Z_n \) gives the amplitudes of CTQW over \( Z_2 \) up to a time scale 2.

B. Dihedral group \( D_{2n} \)

In order to study CTQW on dihedral group, we consider the odd and even \( n \), separately.

1. Odd \( n \). In this case, the \( 1/2(n + 3) \) conjugacy classes are given by

\[
C_0 = \{e\}, \quad C_1 = \{b, ab, \ldots, a^{n-1}b\}, \quad C_2 = \{a, a^{-1}\}, \\
C_3 = \{a^2, a^{-2}\}, \ldots, \quad C_{n+1} = \{a^{n-1}, a^{-n-1}\}.
\]

The character is given by this character table

<table>
<thead>
<tr>
<th>class</th>
<th>1</th>
<th>( a^r ) (1 ( \leq r \leq (n - 1)/2 )</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \chi_0 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_1 )</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( \psi_j ), (1 ( \leq j \leq (n - 1)/2 )</td>
<td>2</td>
<td>( \omega^j + \omega^{-j} )</td>
<td>0</td>
</tr>
</tbody>
</table>

with \( \omega := e^{2\pi i/n} \). Now, by using Eq. (12), we obtain the following amplitudes

\[
\langle 1 | \phi_0(t) \rangle = \frac{1}{n} (\cos nt + n - 1),
\]

\[
\langle a^j b | \phi_0(t) \rangle = -\frac{i}{n} \sin nt, \quad j = 0, 1, \ldots, n - 1,
\]
\[
\langle a^r | \phi_0(t) \rangle = \frac{1}{2n} \left\{ e^{-int} + e^{int} + 2 \sum_{j=1}^{n-1} (\omega^j + \omega^{-j}) \right\}
\]
\[
= \frac{1}{n} \left( \cos nt - 1 \right), \quad r = 1, \ldots, n - 1.
\]

(17)

Now, by considering the normal subgroup \( H = Z_n = \langle a \rangle \), the scalar representations with kernel \( Z_n \) are \( \chi_0 \) and \( \chi_1 \), so that the sum of the amplitudes corresponding to the cosets are obtained as follows

\[
\sum_{l=0}^{n-1} \langle a^l | \phi_0(t) \rangle = \frac{|Z_n|}{|D_{2n}|} \left\{ e^{-in\chi_0(b)t} \chi_0(e) + e^{-in\chi_1(b)t} \chi_1(e) \right\}
\]
\[
= \frac{1}{2} (e^{-int} + e^{int}) = \cos nt,
\]

\[
\sum_{l=0}^{n-1} \langle a^l b | \phi_0(t) \rangle = \frac{|Z_n|}{|D_{2n}|} \left\{ e^{-in\chi_0(b)t} \chi_0(b) + e^{-in\chi_1(b)t} \chi_1(b) \right\}
\]
\[
= \frac{1}{2} (e^{-int} - e^{int}) = -i \sin nt.
\]

As it was expected, the sum of the amplitudes corresponding to the cosets gives the amplitudes of the CTQW over \( D_{2n}/Z_n \cong Z_2 \) up to a scale \( n \) on time.

Now, if we consider the non-normal subgroup \( Z_2 = \langle b \rangle \), then we obtain

\[
\langle 1 | \phi_0(t) \rangle + \langle b | \phi_0(t) \rangle = \frac{1}{n} (e^{-int} + n - 1),
\]
\[
\langle a^l | \phi_0(t) \rangle + \langle a^l b | \phi_0(t) \rangle = \frac{1}{n} (e^{-int} - 1), \quad l = 1, \ldots, n - 1.
\]

As the above calculations show, the amplitudes of CTQW on all cosets \( \{ a^l H, l = 1, \ldots, n - 1 \} \) are the same. If we take the sum of the amplitudes which are equal, we obtain the following amplitudes

\[
\langle 1 | \phi_0(t) \rangle + \langle b | \phi_0(t) \rangle = \frac{1}{n} (e^{-int} + n - 1),
\]
\[
\sum_{l=1}^{n-1} \langle a^l | \phi_0(t) \rangle + \langle a^l b | \phi_0(t) \rangle = \frac{(n - 1)}{n} (e^{-int} - 1),
\]

where the resulting two amplitudes are the same as the amplitudes of the CTQW on the complete graph \( K_n \) up to the phase factor \( e^{it} \).

2. **Even \( n \).** For even \( n = 2m \), the \( m + 3 \) conjugacy classes are given by

\[
C_0 = \{ 1 \}, \quad C_r = \{ a^r, a^{-r} \}; \quad 1 \leq r \leq m - 1, \quad C_m = \{ a^m \},
\]
\[
C_{m+1} = \{ a^{2j} b; \quad j = 0, \ldots, m - 1 \}, \quad C_{m+2} = \{ a^{2j+1} b; \quad j = 0, \ldots, m - 1 \}.
\]
In this case, for the purpose of investigation of CTQW, we must combine some of the conjugacy classes in order for a connected graph to be obtained, i.e. we take the following classes as the new classes:

\[ \hat{C}_0 = \{1\}, \quad \hat{C}_1 = C_{m+1} \cup C_{m+2}, \quad \hat{C}_2 = C_1, \quad \hat{C}_3 = C_2, \ldots, \hat{C}_{l+1} = C_l. \]

Then, the calculation of amplitudes is similar to that of dihedral groups with odd \(n\).

Comparing the above sum of amplitudes with the amplitudes given by Eq. (17) implies that taking the sum of amplitudes associated with the cosets \(Hg\) with \(H = Z_2 = \langle a^m \rangle\), gives the amplitudes of CTQW over the quotient group \(D_{2n}/Z_2 \cong D_n = D_{2m}\).

**C. Generalized quaternion group \(Q_n\)**

For any positive integer \(n\), the generalized quaternion group \(Q_n\) is by definition the discrete group that is generated freely by elements \(a\) and \(b\) as

\[ Q_n = \langle a, b : a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle. \]

Therefore, \(Q_n\) is a finite group of order \(|Q_n| = 4n\) (the ordinary quaternion group corresponds to the case \(n = 2\)). The \(n + 3\) conjugacy classes are given by

\[
C_0 = \{1\}, \quad C_r = \{a^r, a^{-r}\}; \quad 1 \leq r \leq n - 1, \\
C_n = \{a^n\}, \quad C_{n+1} = \{a^2b; \ j = 0, \ldots, n - 1\}, \\
C_{n+2} = \{a^{2j+1}b; \ j = 0, \ldots, n - 1\}. 
\]

The irreducible characters of \(Q_n\) are displayed in the following table:

<table>
<thead>
<tr>
<th>Class</th>
<th>(e) ((1 \leq r \leq n - 1))</th>
<th>(a^n) ((0 \leq j \leq n - 1))</th>
<th>(a^{2j}b) ((0 \leq j \leq n - 1))</th>
<th>(a^{2j+1}b) ((0 \leq j \leq n - 1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\chi_0)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(\chi_1)</td>
<td>1</td>
<td>1</td>
<td>(-1)</td>
<td>(-1)</td>
</tr>
<tr>
<td>(\chi_2)</td>
<td>1</td>
<td>((-1)^r)</td>
<td>((-1)^n)</td>
<td>(i^n)</td>
</tr>
<tr>
<td>(\chi_3)</td>
<td>1</td>
<td>((-1)^r)</td>
<td>((-1)^n)</td>
<td>(-i^n)</td>
</tr>
<tr>
<td>(\psi_{k},)</td>
<td>2</td>
<td>(2\cos(\pi kr/n))</td>
<td>(2(-1)^k)</td>
<td>0</td>
</tr>
</tbody>
</table>

\((1 \leq k \leq n - 1)\)
As in the case of $D_{2n}$ with even $n = 2m$, for the purpose of investigation of CTQW, we must combine some of the conjugacy classes (19) in order for a connected graph to be obtained. To do so, we take the following classes as the new classes:

$$
\hat{C}_0 = \{1\}, \quad \hat{C}_1 = C_{n+1} \cup C_{n+2}, \quad \hat{C}_2 = C_1, \\
\hat{C}_3 = C_2, \ldots, \hat{C}_{l+1} = C_l, \quad l = 1, \ldots, n.
$$

If we choose the normal subgroup $Z_{2n} \cong \langle a \rangle$, then, as in the case of the previous examples, one can evaluate the corresponding sums of the amplitudes of CTQW over $Q_n$ as follows:

$$
\sum_{r=0}^{2n-1} \langle a^r | \phi \rangle = \cos 2nt, \quad \sum_{r=0}^{2n-1} \langle a^r | b \rangle = -i \sin 2nt,
$$

which indicates that CTQW over $Q_n$ induces a CTQW over $Q_n/Z_{2n} \cong Z_2$, up to a time scale equal to $2n$.

Now, we choose the normal subgroup $Z_2 \cong \langle a^n \rangle$. Then, as in the case of the previous examples, one can evaluate the corresponding sums of the amplitudes of CTQW over $G$ as follows:

$$
\langle e | \phi \rangle + \langle a^n | \phi \rangle = \frac{1}{n} (\cos 2nt + n - 1),
$$

$$
\langle a^r | \phi \rangle + \langle a^{n+r} | \phi \rangle = \langle a^{-r} | \phi \rangle + \langle a^{n-r} | \phi \rangle = \frac{1}{n} (\cos 2nt - 1), \quad 1 \leq r \leq n - 1,
$$

$$
\langle a^{2j} | b \rangle + \langle a^{(2j+n)} | b \rangle = \langle a^{(2j+1)} | b \rangle + \langle a^{2j+n+1} | b \rangle = \frac{-i}{n} \sin 2nt, \quad 0 \leq j \leq n - 1.
$$

4. Conclusion

It was shown that CTQW over a group association scheme defined on a finite group $G$ induces a CTQW over group association scheme defined on $G/H$, where $H$ is a normal subgroup of $G$ with prime index. Even though this claim was proved for normal subgroups with prime index, it was seen that for other normal subgroups or even non-normal subgroups, the result is also true.

Appendix A

**Clifford theorem.** Let $H \trianglelefteq G$ be a normal subgroup of prime index $p$, $T = (t_0 = e, t_1, \ldots, t_{p-1})$ a transversal of $G/H$ and $\rho$ an irreducible representation of $H$. Then exactly one of the two following cases applies:

1. If $\rho \cong \rho^i$ for $i = 0, 1, \ldots, p - 1$, where $\rho^i(g) := \rho(t_i^{-1}gt_i)$. Then, the induction decomposes into irreducibles as

$$
(\rho \uparrow_T G)^A = \bigoplus_{i=0}^{p-1} \lambda_i \rho_i.
$$
where $\lambda_i : t_l \to \omega_p^i$ ($\omega_p = e^{2\pi i/p}$ is primitive $p$th root of unity) is a representation of $G/H$, $\hat{\rho}$ is an extension of $\rho$ to $G$, and

$$A = \text{diag}(\hat{\rho}^i | i = 0, 1, \ldots, p - 1) \cdot (\text{DFT}_p \otimes I_{|H|/p}).$$

2. If $\rho \neq \rho^i$ for $i = 0, 1, \ldots, p - 1$. Then, the induction $\rho \uparrow_T G$ is irreducible:

$$(\rho \uparrow_T G) \downarrow H = \oplus_{i=0}^{p-1} \rho^i.$$

References