SOLUTION FOR THE LMG MODEL HAMILTONIAN
BASED ON THE RECURRENCE
FORMULA AND JACOBI MATRIX

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Based on the recursion relation formulas and the Jacobi matrix from the theory of orthogonal polynomials, a numeric solution for the Lipkin–Meshkov–Glick (LMG) model Hamiltonian is given, without considering any restriction on the parameters specifying the strengths of the interactions included in the LMG model. Moreover, via this method, we have studied the shape phase transitions for one example of nuclei. At the end we compare the results thus obtained with those of Bethe ansatz and Hartree–Fock methods.

Keywords: The recurrence formula and Jacobi matrix; strengths of the interactions; shape phase transitions.

1. Introduction

The pairing interaction is the part of the fermion Hamiltonian responsible for the superconducting phase in metals and in nuclear matter or neutron stars. It is also responsible for the pairing in the corresponding finite systems, namely ultrasmall superconducting grains and atomic nuclei.† The main feature of the pairing interaction is that it correlates pairs of particles in time-reversed states. The concept can be traced back to the seniority scheme introduced by Racah in atomic physics‡ and its physical significance was first realized in the study of superconductivity in macroscopic systems by Bardeen, Cooper and Schrieffer.§ In 1958, Bohr Mottelson and Pines¶ observed that the gap in the excitation spectrum of even–even nuclei might be due to correlations between the nucleons in the nucleus that are similar to those between electrons in a superconductor. So many authors have been led
to use the pairing force Hamiltonian, which had been successfully used to describe superconductors, as a model Hamiltonian in nuclear physics but they found that the model gives a fairly good description of nuclear properties, because it is based on wave functions with indefinite number of particles. A number of methods for treating the pairing problem have been previously proposed. Richardson has shown that it is possible to determine simultaneously the eigenvalues and the eigenstate of the Bardeen–Cooper–Schrieffer (BCS) Hamiltonian.\(^5\)-\(^7\) Another model, Lipkin–Meshkov–Glick (LMG) model Hamiltonian, was proposed in 1965 to describe shape phase transition in nuclei,\(^8\)-\(^10\) magnetic properties of molecules,\(^11\) the nuclear giant monopole resonance, interacting bosons in double-well structures\(^12\) and to investigate the role of entanglement in quantum phase transitions (QPT).\(^13\) The LMG model is a toy model that describes, in its simplest version, two shells for the nucleons and an interaction between nucleons in different shells. It has proved to be a traditional testing ground for new approximation techniques,\(^14\),\(^15\) since it is numerically solvable. Solution has been approached by introducing sophisticated mathematical tools. Bethe ansatz method (BAM) is one of the algebraic approaches for deriving solutions of the nuclear pairing problem but is more costly in terms of computational effort than exact diagonalization. In 1998, Pan and Draayer\(^16\) derived a solution of the LMG model by this method only for the case of \(W^2 \geq V^2\) (using the weakly interacting limit solutions). But in 2006, Nishiyama and others\(^17\) obtained solutions of the LMG Model for the most general case including \(W^2 < V^2\).

Here, in this work, by using the Recurrence formula and the Jacobi matrix from the theory of orthogonal polynomials we obtain a solution for the LMG model for all regimes. In BAM when \(n\) increases we achieve polynomials that they are not exact solvable and must be solved numerically. Therefore, both Jacobian recursion method and BAM must be solved numerically (see Ref.\(^17\) where its authors solved LMG Hamiltonian by numerically solving the BAM). Our method is relatively simple and given better accuracy with respect to the BAM. The structure of the paper is as follows: LMG Model and its Jordan–Schwinger fermion–boson mapping are presented in Sec. 2. The Recurrence formula and Jacobi matrix are introduced in Sec. 3 and we will illustrate the behavior of the eigenvalue spectra and the energy gap for the LMG model in this section. Conclusions and discussions are given in Sec. 4.

2. The Model

The LMG model has been extensively used for decades to simulate the phase transition from spherical to deformed shapes in finite nuclei. It is a schematic model describing the scattering of particle hole pairs between two shells of different parity \(\sigma\). To show the effectiveness of our method, we will apply it to this model. In the LMG model, \(N\) particles can distribute themselves on two \(N\)-fold degenerate levels distinguished by a quantum number \(\sigma\) with \(\sigma = \pm 1\). \(a_{p\sigma}^\dagger (q_{p\sigma})\) is the corresponding fermion creation (annihilation) operator for a particle in the state \(p\) of the \(\sigma\) level,
where \( p = 1, 2, \ldots, N \). The Hamiltonian of the LMG model can be written as

\[
H = \varepsilon \sum_{p,p'=1,2,\ldots,N;\sigma=\pm 1} \sigma a^\dag_p \sigma a_{p'} + V \sum_{p,p'=1,2,\ldots,N;\sigma=\pm 1} a^\dag_{p+\sigma} a^\dag_{p'-\sigma} a_{p-\sigma} a_{p'-\sigma} + W \sum_{p,p'=1,2,\ldots,N;\sigma=\pm 1} a^\dag_{p\sigma} a^\dag_{p'-\sigma} a_{p'-\sigma} a_{p\sigma},
\]

(1)

where \( V \) and \( W \) are parameters specifying the strengths of the interactions. The interaction term \( V \) scatters a pair of particles across the Fermi level, i.e. it is a two-particle hole interaction, while the term \( W \) exchange particles in the two levels. By introducing the so-called pseudo spin operators

\[
J_+ = \sum_p a^\dag_p a_{p-}, \quad J_- = \sum_p a^\dag_{p-} a_p, \quad J_z = 1/2 \sum p \sigma \sigma a^\dag_{p\sigma} a_{p\sigma},
\]

(2)

Eq. (1) can be written as

\[
H = \sum_\sigma \varepsilon_\sigma n_\sigma + V(J_+^2 + J_-^2) + 2W(c_2 - J_z^2),
\]

(3)

where, \( c_2 \) is the Casimir operator of the pseudo spin algebra \( SU(2) \) and

\[
n_\sigma = \sum_p a^\dag_{p\sigma} a_{p\sigma}.
\]

(4)

Let us introduce the following Jordan–Schwinger fermion-boson mapping,

\[
J_+ \Rightarrow b^\dag_+ b_-, \quad J_- \Rightarrow b^\dag_- b_+, \quad J_z \Rightarrow 1/2(b^\dag_+ b_+ - b^\dag_- b_-), \quad n_\sigma \Rightarrow N_\sigma = b^\dag_\sigma b_\sigma,
\]

(5)

where \( b^\dag_\sigma (b_\sigma) \) with \( \sigma = + \) or \( - \) are boson creation (annihilation) operators. Then by using this mapping, the Hamiltonian (2) and (3) can be written as

\[
H = \sum_\sigma \varepsilon_\sigma N_\sigma + V(b^\dag_+ b^- + b^\dag_- b^+ + \frac{1}{2}(b^\dag_+ b_+ - b^\dag_- b_-)) + 2W(c_2 - 1/4N^2) + 2WN_+ N_-,
\]

(6)

with

\[
N_\sigma = b^\dag_\sigma b_\sigma, \quad \sigma = \pm
\]

(7)

so that, the Casimir operator is written as

\[
c_2 = 1/4N(N + 2),
\]

(8)

where

\[
N = \sum_\sigma N_\sigma.
\]

(9)

By using the pseudo spin algebra, the LMG Hamiltonian can be written as

\[
H = \varepsilon J_z + \frac{1}{2} V(J_+^2 + J_-^2) + \frac{1}{2} W(J_+ J_- + J_- J_+).
\]

(10)
It should be noticed that the term \((J_+^2 + J_-^2)\) contributes to the phase transition, and the term \((J_+ J_- + J_- J_+)\) does also if \(W < 0\), but tends to suppress the transition if \(W > 0\).\(^{18}\) In dimensionless form, the Hamiltonian thus reduces to

\[ H = J_z + \frac{\lambda}{2N}(J_+^2 + J_-^2). \]  

For \(W = 0\), this model has been studied numerically,\(^8\,^{10}\) where the \(2j + 1 = (N + 1)\)-dimensional representations of the SU(2) operators have been used. Here the interaction is scaled by \(N\) to ensure that \(H\) is extensive, the operators \(J_+^2\) and \(J_-^2\) effectively scale as \(N^2\) and define the dimensionless coupling constant \(\lambda\) as \(\lambda = NV/\varepsilon\). The phases can be characterized further by the energy gap

\[ \Delta_n = E_{n+1} - E_n. \]  

3. Construction of Eigenstates by Using the Recurrence Formula

If we act the Hamiltonian \(H\) on the unit vectors \(|\phi_i\rangle\), \(i = 0, 1, \ldots, n\), as an orthonormal basis in which the Hamiltonian \(H\) is tridiagonal, we can obtain

\[ H|\phi_i\rangle = \beta_i+1|\phi_i+1\rangle + \alpha_i|\phi_i\rangle + \beta_i|\phi_{i-1}\rangle, \quad i = 0, 1, \ldots, n \]  

in order to obtain the energy spectrum, we use the recursion relations in Ref. 27

\[ xQ_k(x) = \beta_{k+1}Q_{k+1}(x) + \alpha_{k+1}Q_k(x) + \beta_kQ_{k-1}(x), \quad k \geq 1. \]  

Recursion relations give our polynomials that we can obtain on the other hand by

\[
x \begin{pmatrix} Q_0(x) \\ Q_1(x) \\ Q_2(x) \\ \vdots \\ Q_{n-1}(x) \\ Q_n(x) \end{pmatrix} = \begin{pmatrix} \alpha_0 & \beta_0 & 0 & 0 & \cdots & 0 \\ \beta_0 & \alpha_1 & \beta_1 & 0 & \cdots & 0 \\ 0 & \beta_1 & \alpha_2 & \beta_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha_{n-1} \end{pmatrix} \begin{pmatrix} Q_0(x) \\ Q_1(x) \\ Q_2(x) \\ \vdots \\ Q_{n-1}(x) \\ Q_n(x) \end{pmatrix}
\]

\[
+ \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \cdot x \begin{pmatrix} \beta_{N-1}Q_N(x) \end{pmatrix}
\]

If we choose \(x\) to be a zero of \(Q_N(x)\), say \(x = x_i\), then we obtain \(x_iQ(x_i) = JQ(x_i)\), where \(J\) is called the Jacobi matrix and the eigenvalues \(x_1, \ldots, x_N\)
of that are the zeros of $Q_N(x)$ and the eigenvector corresponding to $x_i$ is $(Q_0(x_i), Q_1(x_i), \ldots, Q_{N-1}(x_i))^T$.\(^{28}\) If we choose $x = x_i$ then the end matrix will be zero; therefore, the coefficients $\alpha$ and $\beta$ are equal to the coefficients in the recursion relations. Equations (2)–(6) can be written as

$$H = (\varepsilon + N_+ + \varepsilon - N_-) + V(J^2_+ + J^2_-) + WN + 2WN_+N_-.$$  \hspace{1cm} (16)

We assume that $N$ identical nucleons occupy two single-particle orbitals $\varepsilon_1 = -\varepsilon$ and $\varepsilon_2 = \varepsilon$, and consider unit vectors $|\phi_i\rangle$, denote the states of $H$, with two representations that are given by

$$|\phi_i\rangle = \begin{cases} |J, J - 2k_i\rangle & \text{for } k_i = 0, 1, \ldots, J, \\
|J, J - (2k_i + 1)\rangle & \text{for } k_i = 0, 1, \ldots, J - 1. \end{cases} \hspace{1cm} (17)$$

As an orthonormal basis in which the Hamiltonian $H$ is tridiagonal, i.e. we have

$$H|\phi_i\rangle = \beta_{i+1}|\phi_{i+1}\rangle + \alpha_i|\phi_i\rangle + \beta_i|\phi_{i-1}\rangle, \hspace{1cm} i = 0, 1, \ldots, n. \hspace{1cm} (18)$$

The action of $H$ on the first unit vectors $|J, J - 2k_i\rangle$, gives

$$H|J, J - 2k_i\rangle = (2\varepsilon(J - 2k_i) + 2WJ + 8Wk(J - k_i))|J, J - 2k_i\rangle$$

$$+ \hbar^2V\sqrt{4k_i(2k_i - 1)(2(J - k_i) + 1)(J - k_i + 1)}|J, J - 2k_i + 2\rangle$$

$$+ \hbar^2V\sqrt{4(J - k_i)(2J - 2k_i - 1)(k_i + 1)(2k_i + 1)}|J, J - 2k_i - 2\rangle. \hspace{1cm} (19)$$

Therefore, the coefficients $\alpha_{k_i}$ and $\beta_{k_i}$ are given by

$$\alpha_{k_i} = 2\varepsilon(J - 2k_i) + 2WJ + 8Wk(J - k_i),$$

$$\beta_{k_i} = \hbar^2V\sqrt{4k_i(2k_i - 1)(2(J - k_i) + 1)(J - k_i + 1)}.$$

For the second unit vectors $|J, J - (2k_i + 1)\rangle$, we have

$$H|J, J - (2k_i + 1)\rangle = (2\varepsilon(J - 2k_i - 1) + 2W(J + (2k_i + 1)(2J - 2k_i - 1))|J, J - (2k_i + 1)\rangle$$

$$+ \hbar^2V\sqrt{4k_i(J - k_i)(2J - 2k_i + 1)(2k_i + 1)}|J, J - (2k_i + 1) + 2\rangle$$

$$+ \hbar^2V\sqrt{(2J - 2k_i - 1)(2k_i + 2)(2k_i + 3)}|J, J - (2k_i + 1) - 2\rangle. \hspace{1cm} (21)$$

Therefore, the coefficients $\alpha_{k_i}$ and $\beta_{k_i}$ are given by

$$\alpha_{k_i} = (J - 2k_i - 1) + 2W(J + (2k_i + 1)(2J - 2k_i - 1)),$$

$$\beta_{k_i} = \hbar^2V\sqrt{4k_i(J - k_i)(2J - 2k_i + 1)(2k_i + 1)}.$$

The energy gap between the first excited state and the ground state is obtained as

$$\Delta = E_2 - E_1. \hspace{1cm} (23)$$

The pairing gap, which measures the “binding energy” of pairs of nucleons moving in time-reversal states and forming Cooper pairs is order parameter. The order
parameter measures the magnitude of the distortion. The pairing order parameter can also be subjected to a macroscopic change and the system undergoes a phase transition from the normal to the superconducting state.\textsuperscript{19}

4. Conclusions and Discussions

In this paper we got a solution for the LMG model Hamiltonian, and energy gap between the first excited state and the ground state, as are shown in Fig. 1. We mentioned the gap at $\lambda = 1$, $V = 0.1$ which is positive in the first phase, normal nuclei, and vanishes like $\sqrt{1-\lambda^2}$ as $\lambda$ approaches 1, many-body interacting quantum system as well as one of the simplest to show a quantum transition in the regime of strong coupling. This transition is by now well-understood: the ground state becomes degenerate and a macroscopic change in the ground state energy takes place. Furthermore, at the transition value of the coupling, the density of states diverges at the ground state energy as the number $N$ of interacting particles becomes large. The phase at $\lambda < 1$ is called the normal phase while the symmetry (parity) breaking phase at $\lambda > 1$ is called the deformed phase. Recent calculations for nonzero values of $\lambda$ cause particle-hole excitations across the gap, and at $\lambda = \pm 1$ the model exhibits a phase transition from an unreformed first phase to a deformed second phase. For $\lambda \geq 1$ this phase transition is confined to the lower part of the spectrum in that higher up in the spectrum the normal phase still prevails. In fact, for a specific $\lambda > 1$ there is an energy $E_k$, such that the ordered spectrum $E_k$ is for $k < k_c$ associated with the deformed phase, while for $k > k_c$ the energies relate to the normal phase.\textsuperscript{20} In other words, the transition moves up in the spectrum from

![Fig. 1. Shows $\Delta$ as a function of the parameter $V$ for 10 particles, $W = 0$.](image-url)
the ground state $E_1$ to $E_{N/2}$ when it increases from unity to infinity (we confine ourselves to the lower half of the spectrum as the symmetry of the model yields the same pattern in the upper half but mirror reflected). For finite $N$, $E_{k_c}$ is clearly characterized by the minimum of the level distance; this minimum gap vanishes for $N \to \infty$. In fact, at $\lambda = 1$ the large $N$ scaling behavior$^{21,22}$ is found

Fig. 2. Shows $E$ as a function of the parameter $V$ for 10 particles, $W = V$.

Fig. 3. Shows $E$ as a function of the parameter $V$ for 10 particles, $W = 0$. 

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Fig. 4. Shows $E$ as a function of the parameter $V$ for 10 particles, $W = 0.5V$.

Fig. 5. Shows $E$ as a function of the parameter $V$ for 10 particles, $W = 2V$.

\[
\Delta E_k \sim \left( \frac{k}{N} \right)^{1/3},
\]

whereas for $\lambda > 1$ the gap around $E_{k_c}$ behaves as

\[
\Delta E = \frac{2\pi \sqrt{\lambda^2 - 1}}{\ln N}.
\]
This corresponds to the high density of states. We mention that a mean-field approach yields the result

$$E_k(\lambda) = \begin{cases} 
    k \sqrt{1 - \lambda^2} & \text{for } \lambda < 1, \\
    k \sqrt{2(\lambda^2 - 1)} & \text{for } \lambda > 1. 
\end{cases} \quad (26)$$

For low energies $E < E_c(\lambda)$, the states were deformed, the order parameter was nonzero and an odd–even degeneracy was observed. For $E > E_c(\lambda)$, all these phenomena disappeared and a normal regime, similar to $\lambda < 1$ was recovered.\(^{23}\) The transition region between those two regimes had the typical signature of high density of states. Being obviously unable to reproduce the richer structure described above, however, the last two equations do describe the lower part of the spectrum remarkably well. We can see the behavior of transition from normal nuclear behavior to deformed behavior. Figures 2–6 show the critical point of the pairing transition from normal nuclear behavior to deformed behavior. In these critical points, there is a QPT. The QPT describes structural change of the ground state energy spectrum of many-body systems associated with the variation of coupling parameters and there can be a level-crossing where the excited state and ground state levels interchange at the critical point, creating nonanalyticity of the ground state energy which is a function of the coupling parameter. The quantum solutions exhibit both crossings between levels with different parities and anticrossings (level repulsion) between states with the same parities. There are two types of phase transition in nuclei: the phase transition to superfluidity and to deformation. The first one is easy to trace with the standard BCS theory which gives the critical condition.\(^{24}\) Many physical systems (nuclei, molecules, atomic clusters, etc.) are characterized in their
equilibrium configuration by a shape. These shapes are in many cases rigid. However, there are several situations in which the system is rather floppy and undergoes a phase transition between two different shapes. A challenging problem is how to describe properties of the system in the phase transition region and in particular at the phase transition point. The nuclear deformation plays a crucial role in determining their properties such as quadrupole moment, nuclear size and isotope shift. The second one, the transition to deformation, takes place as some control parameters vary along an isotopic (isotonic) chain, which, for example, brings the system from a spherical to a deformed region. However, shape transitions in the Lipkin model are shown to exhibit the same features, indicating that this is a direct consequence of parity projection in finite, mixed parity systems independent of the nature of interacting particles. The second type of phase transition is also called shape phase transition. We can see as a function of increasing $W/V$ high-lying levels become degenerate more quickly than the low-lying ones; it follows from this that the critical value of $W/V$ for the phase transition is level-dependent. We can see that there occur many remarkable crossings of the eigenvalues among different levels and occurs at stronger interaction regimes as the ratio $W/V$ increases and suggests the existence of a new phase transition. Crossing cannot be found in the case of $W^2 < V^2$. We can obtain results for all regimes. One needs a program in order to find polynomials with respect to alpha, beta and solve them, sort their roots and plot them finally. We write our program with MATLAB 7.1 software and consider 5 odd $n$ and 6 even $n$. Figures 7 and 9 compare $E$ as a function of $V$ (by considering $W = 0$) obtained from the spectral method with those of Bethe

![Graph](attachment:image.png)

Fig. 7. Shows $E$ as a function of the parameter $V$ for 10 particles, $W = 0$, upper level for both Bethe ansatz and spectral method.
Fig. 8. Shows $E$ as a function of the parameter $V$ for 10 particles, $W = 0$, lower level for both Bethe ansatz and spectral method.

Fig. 9. Shows $E$ as a function of the parameter $\chi = (V/\varepsilon)(\Omega - 1)$ for 10 particles, $W = 0$ lower level for both Hartree–Fock and spectral method.

ansatz and Hartree–Fock methods, respectively; Fig. 8 compares the lower levels obtained from Bethe ansatz and spectral methods. Obviously, for finding upper levels we should consider more polynomials that have higher degree and therefore solving them takes more time and more CPU, but in computer with capacity Intel core 2 Duo 2.2 GHZ, Win XP and Service pack 3, it takes three minutes or less in the process with increasing number of particles. The LMG model was conceived
as a test model in nuclear physics. It is simple enough to be solved, but it is yet nontrivial. For that reason, since it was established has been used to validate many fermion approximation methods like Hartree–Fock.\textsuperscript{18}

References