Investigation of shape phase transition in the $U(5) \leftrightarrow SO(6)$ transitional region by catastrophe theory and critical exponents of some quantities

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In this paper, we have analyzed the critical behavior of even–even Ru and Pd isotopes between $U(5)$ and $SO(6)$ limits of interacting boson model via Catastrophe Theory in combination with a coherent state formalism to generate energy surfaces. The parameters of the Hamiltonian are determined via least-square fitting to the experimental data for different Ru and Pd isotopes. Our results suggest a second-order phase transition in these isotopic chains and propose the best candidates for $E(5)$ critical symmetry. Also, the analogy between the critical exponents of ground state quantum phase transition and Landau values for the critical exponents of thermodynamic phase transitions are described.

Keywords: Catastrophe theory; coherent states; interacting boson model; energy surface; phase transition point; bifurcation set; critical exponents.

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1. Introduction

Drastic changes in the properties of physical systems are called phase transitions. For systems which are characterized by order parameters, phase transitions occur as some of the parameters, called control parameters, which constrain the systems, are varied. Temperature-governed phase-transitions in which the control parameter is the temperature $T$ have been known for many years.1 Gilmore et al.2 introduced phase transitions in which the control parameter $g$ is a parameter appearing in the quantum Hamiltonian that describes the system. The concept of a quantum phase transition,3–6 sometimes called zero temperature or ground state phase transitions, refers to a sudden transformation in the structure of the ground state as a function of some variable. The term quantum phase transition originates from condensed-matter physics, where it was introduced for transitions between specific ordered and disordered phases.3–5 Because of vanishing thermal fluctuation,
the only responsible motions for the onset of disorder are the quantum fluctuation. The study of quantum phase transitions in finite nuclear systems has recently been the subject of many investigations. In quantum systems, however, phase transitions can take place at zero temperature and the change in order is invoked by some other parameters. In its ground state, an atomic nucleus has a stable geometrical shape that results from the interaction among its nucleons. As the number of nucleons changes from nucleus to nucleus, shape phase transitions occur, in which the geometrical configuration changes. The order parameter is the shape of the nucleus. In nuclear physics, quantum phase transitions can be studied most easily using algebraic techniques that associate a specific mathematical symmetry with the different nuclear shapes. The Interacting Boson Model (IBM) was proposed in 1975 by Iachello and Arima to describe collective excitations of atomic nuclei. This model combined the ingredients of the two paradigms which are used in nuclear physics at that time, i.e., the shell model and the collective model. Nucleons in an even–even isotope are divided into an inert core and an even number of valence particles. These particles are then considered as coupled into two kinds of bosons that may carry either a total angular momentum 0 or 2, and are respectively called the $s$ - and $d$-bosons. A low-lying collective state of an even–even nucleus with $2N$ valence nucleons is approximated as an $N$-boson state and $N$(total boson numbers) is conserved. Because of the close equality of the masses and of the interactions between nucleons, the Hamiltonian of the nucleus is (approximately) invariant with respect to transformations between neutron and proton states. The bilinear operator that may be formed with the $s$ - and $d$-boson creation and annihilation operators close into the $U(6)$ algebra, whose three possible subgroup chain match the $U(5), SU(3)$ and $SO(6)$ solution of the Bohr Hamiltonian, i.e., spherical, axially deformed and $\gamma$-unstable shapes, respectively. One of the most important features of the IBM is the existence of dynamical symmetries, each represents a phase of nuclear collective motion. The symmetries of the IBM and the transitions between them are illustrated in Fig. 1, i.e., Casten triangle. Recently, the concept of critical-point symmetries is introduced to describe nuclei at the phase-transitional point. The importance of the critical point symmetries lies in the fact that they provide a classification of states and analytic expressions for observables in regions where the structure changes most rapidly. These are not dynamical symmetries with group theoretical roots, but geometrical descriptions, based on potentials with very simple shapes. Critical symmetry $E(5)$ corresponds to the transition from spherical to $\gamma$ unstable shapes.

In this paper, we used a $U(5) \leftrightarrow SO(6)$ transitional Hamiltonian according to an affine $SU(1, 1)$ algebra. In the next sections, after a brief presentation of the coherent states, we drive energy surfaces. Next, we introduce a catastrophe theory and then we analyze the critical behavior of even–even isotopic chains of Ru and Pd. Then, after a brief review of Landau theory, we determine the critical exponents of ground state quantum phase transition. Finally, the last section is for summarizing.
2. $U(5) \leftrightarrow SO(6)$ Transitional Hamiltonian

The $SU(1,1)$ Algebra has been described in detail in Refs. 27 and 28. Here, we briefly outline the basic ansatz and summarize results. The Lie algebra corresponding to the $SU(1,1)$ group is generated by $S^\nu$, $\nu = 0$ and $\pm$, which satisfies the following commutation relations:

$$[S^0, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = -2S^0. \quad (1)$$

The Casimir operator of $SU(1,1)$ can be written as

$$\hat{C}_2 = S^0(S^0 - 1) - S^+S^-. \quad (2)$$

In IBM, the generators of $d$-boson pairing algebra created by

$$S^+(d) = \frac{1}{2}(d^\dagger \cdot d), \quad S^-(d) = \frac{1}{2}(\bar{d} \cdot \bar{d}), \quad S^0(d) = \frac{1}{4} \sum_\nu (d^\dagger_\nu d_\nu + d_\nu d^\dagger_\nu). \quad (3)$$

Similarly, $s$-boson pairing algebra forms another $SU^*(1,1)$ algebra generated by

$$S^+(s) = \frac{1}{2}s^\dagger s, \quad S^-(s) = \frac{1}{2}s^2, \quad S^0(s) = \frac{1}{4}(s^\dagger s + ss^\dagger). \quad (4)$$

On the other hand, infinite dimensional $SU(1,1)$ algebra is generated by using

$$S^\pm_n = c_s^{2n+1} S^\pm(s) + c_d^{2n+1} S^\pm(d), \quad S^0_n = c_s^{2n} S^0(s) + c_d^{2n} S^0(d), \quad (5)$$

where $c_s$ and $c_d$ are real parameters and $n$ can be 0, 1, 2, ... These generators satisfy the commutation relation

$$[S^0_n, S^\pm_m] = \pm S^\pm_{m+n}, \quad [S^+_m, S^-_n] = -2S^0_{m+n+1}. \quad (6)$$
Then, \( \{ S^\mu_m, \mu = 0, +, -, \pm 1, \pm 2, \ldots \} \), generates an affine Lie algebra \( SU(1, 1) \) without central extension. With employing the generators of \( SU(1, 1) \) algebra, the following Hamiltonian is constructed for transitional region between \( U(5) \leftrightarrow SO(6) \) limits\(^{27,28}\):

\[
\hat{H} = g S^+_0 S^-_0 + \alpha S^0_1 + \gamma \hat{C}_2(SO(5)) + \delta \hat{C}_2(SO(3)).
\]

\( g, \alpha, \gamma \) and \( \delta \) are real parameters where \( \hat{C}_2(SO(3)) \) and \( \hat{C}_2(SO(5)) \) denote the Casimir operators of these groups. It can be seen that, Hamiltonian (7), would be equivalent with \( SO(6) \) Hamiltonian if \( c_s = c_d \) and with \( U(5) \) Hamiltonian when \( c_s = 0 \) and \( c_d \neq 0 \). Therefore, the \( c_s \neq c_d \neq 0 \) requirement just corresponds to \( U(5) \leftrightarrow SO(6) \) transitional region. In our calculation, we take \( c_d = 1 \) constant value and \( c_s \) vary between 0 and \( c_d \). Consequently, the critical point for this transition is given by \( c_s \approx 0.5 \). Now, if we use the algebraic definition of different terms in Hamiltonian (7), the final form is yield as

\[
H = \frac{g}{4} \left( c^2_s s^\dagger s s + c_s c_d s^\dagger d^\dagger (d \cdot \bar{d}) + c_s c_d (d^\dagger \cdot d^\dagger) s s + c^2_d (d^\dagger \cdot d^\dagger) (d \cdot \bar{d}) \right)
\]

\[
+ \alpha \left( \frac{c^2_s}{4} (s^\dagger s + s s^\dagger) + \frac{c^2_d}{4} \sum_\mu d^\dagger_\mu d_\mu + d_\mu d^\dagger_\mu \right)
\]

\[
+ \gamma \hat{C}_2(SO(5)) + \delta \hat{C}_2(SO(3)).
\]

3. Energy Surfaces

We can investigate the geometric configuration of the considered model in the framework of coherent state. The coherent state formalism of IBM\(^ {29–31}\) connects the algebraic and geometric descriptions of three dynamical symmetry limits and also allows the study of transitions among them. By using this formalism, one can evaluate the ground state energy as a function of shape variables \( \beta \) and \( \gamma \), i.e., deformation parameters,\(^ {35} \) similar to what have been done for \( U(5) \leftrightarrow SO(6) \) and \( U(5) \leftrightarrow SU(3) \) phase transitions.\(^ {36,37} \) The classical limit corresponding to Hamiltonian (8) is obtained by considering its expectation value in the coherent state\(^ {29–31} \)

\[
|N, \alpha_m\rangle = \left( s^\dagger + \sum_m \alpha_m d^\dagger_m \right)^N |0\rangle,
\]

where \( |0\rangle \) is the boson vacuum state, \( s^\dagger \) and \( d^\dagger \) are the boson operators of the IBM, and parameter \( \alpha_m \) can be related to the deformation collective parameters\(^ {35} \)

\[
\alpha_0 = \beta \cos \gamma, \quad \alpha_{\pm 1} = 0, \quad \alpha_{\pm 2} = \frac{\beta}{\sqrt{2}} \sin \gamma.
\]

Energy surface would determine by means of

\[
E = \frac{\langle N, \alpha_m | H | N, \alpha_m \rangle}{\langle N, \alpha_m | N, \alpha_m \rangle}.
\]
Then, the energy surfaces from each part of transitional Hamiltonian can be written as

\[
\langle g S_0^+ S_0^- \rangle = \frac{g}{4} \left( \frac{N(N-1)}{(1+\beta^2)^2} \right) \left( c_s^2 + 2c_sc_d\beta^2 + c_d^2\beta^4 \right), \quad (12a)
\]

\[
\langle \alpha S_0^z \rangle = \frac{\alpha c_s^2}{4} \left( \frac{2N}{1+\beta^2} + 1 \right) + \frac{\alpha c_d^2}{4} \left( \frac{2N\beta^2}{1+\beta^2} + 5 \right), \quad (12b)
\]

\[
\langle \gamma \hat{C}_2(SO(5)) \rangle = \frac{2\gamma N\beta^2}{1+\beta^2}, \quad (12c)
\]

\[
\langle \delta \hat{C}_2(SO(3)) \rangle = \frac{3\delta N\beta^2}{5(1+\beta^2)}. \quad (12d)
\]

These yields the energy surfaces as

\[
E(\beta, \gamma) = \frac{g}{4} \left( \frac{N(N-1)}{(1+\beta^2)^2} \right) \left( c_s^2 + 2c_sc_d\beta^2 + c_d^2\beta^4 \right) + \frac{\alpha c_s^2}{4} \left( \frac{2N}{1+\beta^2} + 1 \right) + \frac{\alpha c_d^2}{4} \left( \frac{2N\beta^2}{1+\beta^2} + 5 \right) + \frac{2\gamma N\beta^2}{1+\beta^2} + \frac{3\delta N\beta^2}{5(1+\beta^2)}. \quad (13)
\]

4. Catastrophe Theory

We are going to analyze the energy surfaces within the catastrophe theory formalism,\(^8\) which is based on important mathematical results of functional analysis. The catastrophe theory, initiated by Thom in the mid 1970’s, has been broadly advertised among all kinds of scientists and technicians. In short, the theory deals with systems in which “continuous causes” can lead to “discontinuous effects” and attempts to study how the qualitative nature of the solutions of equations depends on the parameters that appear in the equations, and provides the appropriate method for modeling the systems that are associated with the sudden changes. Its application in quantum physics was pioneered by Gilmore who showed that the catastrophe theory describes and classifies nonanalytic evolutions of the ground state properties of some many-body systems. Catastrophe theory method allows the definition of the essential parameters to classify the shape and the stability of the energy surface and reduce the number of relevant (or essential) parameters to just two and study all of phase transitions. The application of the catastrophe theory in the interacting boson model was first outlined by Feng et al.\(^10\) and later elaborated by López-Moreno and Castaños.\(^38\) Catastrophe theory allows one to classify phase transitions and to decide if a nucleus is close to criticality. We start by evaluating the critical points of the energy surfaces. For the variable \(\beta\), one finds the algebraic equation

\[
\frac{\partial E}{\partial \beta} = \frac{\beta}{(1+\beta^2)^2} \left( gN(N-1)(c_s + c_d)(c_d - c_s)\beta^2 \right.
\]

\[
+ 2 \left( \frac{N}{2}\alpha c_s^2 + 2\gamma N + \frac{3}{5}\delta N - \frac{N}{2}\alpha c_s^2 \right) (1+\beta^2) \right), \quad (14)
\]
to determine the critical points. From this expression, it is immediate that the \( \beta = 0 \) is a critical point for any values of the parameters of the energy surfaces. For these reasons it is the fundamental root. The Taylor series expansion of the energy surfaces around this fundamental root is given by

\[
E(\beta) = \frac{9}{4} N(N - 1)c_s^2 + \frac{N}{2} o^2 + \frac{1}{4} c_s^2 + 5c_d^2 \]

\[
+ \frac{1}{2} \left[ N(N - 1)gc_s(c_d - c_s) + N \left( \alpha(c_d^2 - c_s^2) + \frac{6}{5} \delta + 4 \gamma \right) \right] \beta^2
\]

\[
+ \left[ \frac{3}{4} N(N - 1)gc_s^2 - N(N - 1)gc_s c_d + \frac{1}{4} N(N - 1)gc_d^2 \right]
\]

\[
+ \frac{1}{2} N \alpha(c_d^2 - c_s^2) - \frac{3}{5} N \delta - 2N \gamma \right] \beta^4 + O(5) + \cdots. \tag{15}
\]

Or can be rewritten in the form

\[
E(\beta) = A + A' \beta^2 + A'' \beta^4 + \cdots. \tag{16a}
\]

While the coefficients are given by

\[
A = \frac{9}{4} N(N - 1)c_s^2 + \frac{N}{2} o^2 + \frac{1}{4} c_s^2 + 5c_d^2, \tag{16b}
\]

\[
A' = \frac{1}{2} \left[ N(N - 1)gc_s(c_d - c_s) + N \left( \alpha(c_d^2 - c_s^2) + \frac{6}{5} \delta + 4 \gamma \right) \right], \tag{16c}
\]

\[
A'' = \frac{3}{4} N(N - 1)gc_s^2 - N(N - 1)gc_s c_d + \frac{1}{4} N(N - 1)gc_d^2
\]

\[
+ \frac{1}{2} N \alpha(c_d^2 - c_s^2) - \frac{3}{5} N \delta - 2N \gamma. \tag{16d}
\]

On the other hand, the bifurcation set is the locus of the points in the space of control parameters at which a transition occurs from one local minimum to another and are obtained by the condition \( \text{det}(H) = 0 \) where \( H \) is the matrix of the second derivatives of the energy surface evaluated at the critical point and became \( \partial^2 E/\partial \beta^2 = 0 \) in the case of a function of one variable. Imposing this condition, one gets the expression

\[
c_s = \frac{g(N - 1)c_d + \sqrt{g^2(N - 1)^2c_d^2 + 4[g(N - 1) + \alpha][\alpha c_d^2 + 4N + \frac{6}{5} \delta]}}{2g(N - 1) + \alpha}. \tag{17}
\]

The Hamiltonian (7) contains two standard dynamical symmetries, \( U(5) \) limit governed with \( c_s = 0 \) and \( c_d \neq 0 \), and \( SO(6) \) limit describe by \( c_s = c_d \). For our purpose in this work, \( c_d \) has been set to unity, i.e., \( c_d = 1 \). Thus, considered Hamiltonian describes \( U(5) \leftrightarrow SO(6) \) transition by varying only the parameter \( c_s \) between 0, i.e., \( U(5) \) limit and 1, namely \( SO(6) \) limit. Consequently, the critical point for this transition is given by \( c_s \geq 0.5 \).
5. Results for Ru and Pd Isotopes

In this section, we have analyzed the critical behavior between $U(5)$ and $SO(6)$ limits. In particular, even–even isotopes of Ru and Pd are considered. These nuclei are well-known examples of the $U(5) \leftrightarrow SO(6)$ transition and can be experimentally verified by, for example, the evolution of the $R_{\text{1/2}}$ ratio, or the evolution of the quadrupole moment. Early work indicated that the neutron-rich Ru and Pd isotopes are well described by this transitional class. More recently, the Ru isotopes were reanalyzed by Frank et al. to locate the second-order critical point. Using two neutron separation energies, level energy systematic, and $B(E2)$ values, these authors concluded that the chain of Ru isotopes in between $^{98}$Ru and $^{110}$Ru can be described as a $U(5) \leftrightarrow SO(6)$ transition with the critical point situated at $^{104}$Ru.

We have extracted the best set of transitional Hamiltonian’s parameters, i.e., $\gamma$, $g$, $\alpha$ and $\delta$, via least-square fit to the available experimental data for excitation energies of selected states, $0^+, 2^+, 4^+, 6^+, 0^+, 2^+, 4^+, 6^+$, $0^+, 2^+, 4^+, 6^+$ (although not all of them are available for all the considered nuclei) or two neutron separation energies. To optimize the Hamiltonian parameters set, we then repeat this procedure, with different values of $c$ and $\varepsilon$, to minimize the root mean square (rms) deviation $\sigma$, between the calculated energy spectra and experimental counterparts which explore the quality of extraction processes. The deviation is defined by the equality

$$\sigma = \left( \frac{1}{N_{\text{tot}}} \sum_{i=1}^{N_{\text{tot}}} \left| E_{\text{exp}}(i) - E_{\text{cal}}(i) \right|^2 \right)^{1/2},$$

where $N_{\text{tot}}$ is the number of energy levels which is included in extraction procedure. Estimated results for Hamiltonian’s quantities which describe Ru isotopes are presented in Table 1.

By using these quantities in into Eq. (17), we can obtain control parameter, $c_s$, for each isotopes of Ru which is represented in Table 2. Also, rms deviations for each isotope which describe the quality of our estimation are also displayed in Table 2. A comparison between theoretical prediction for energy spectra and experimental counterparts of $^{102-104}$Ru isotopes are presented in Fig. 2.

Our results in Table 2 propose $^{104}_{44}$Ru isotope with $c_s = 0.38$, as a candidate for $U(5)$ dynamical symmetry limit and $^{110}_{44}$Ru with $c_s = 0.63$ might be considered as good candidate for $SO(6)$ symmetry limit. Also, $^{104}_{44}$Ru isotope with $c_s = 0.52$, can suggest to be a typical nucleus with $E(5)$ critical point symmetry, e.g., critical

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>$c_d$</th>
<th>$g$</th>
<th>$\alpha$</th>
<th>$\gamma$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ru</td>
<td>1</td>
<td>1</td>
<td>4.4770</td>
<td>-1.2451</td>
<td>-0.0237</td>
</tr>
</tbody>
</table>

Table 1. The parameters of the Hamiltonian determined by least-square fitting to the experimental data (Refs. 42–44) for different Ru isotopes. $\alpha$, $\gamma$ and $\delta$ parameters are in keV.
Table 2. Control parameters $c_s$ for different Ru isotopes, obtained via Eq. (17). $N$ describes total boson number. $\sigma$ factors are given in keV.

<table>
<thead>
<tr>
<th>Isotope</th>
<th>$^{100}\text{Ru}$</th>
<th>$^{102}\text{Ru}$</th>
<th>$^{104}\text{Ru}$</th>
<th>$^{106}\text{Ru}$</th>
<th>$^{108}\text{Ru}$</th>
<th>$^{110}\text{Ru}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td>$c_s$</td>
<td>0.38</td>
<td>0.46</td>
<td>0.52</td>
<td>0.56</td>
<td>0.60</td>
<td>0.63</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>83</td>
<td>75</td>
<td>69</td>
<td>80</td>
<td>91</td>
<td>88</td>
</tr>
</tbody>
</table>

Table 3. The parameters of the Hamiltonian which are determined by least-square fitting to the experimental data (Refs. 42–44) for different Pd isotopes. $\alpha$, $\gamma$ and $\delta$ parameters are in keV.

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>$c_d$</th>
<th>$g$</th>
<th>$\alpha$</th>
<th>$\gamma$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pd</td>
<td>1</td>
<td>1</td>
<td>4.0218</td>
<td>-1.0013</td>
<td>0.0147</td>
</tr>
</tbody>
</table>

Table 4. Control parameters $c_s$ for different Pd isotopes, obtained via Eq. (17). $N$ describes total boson number. $\sigma$ factors are given in keV.

<table>
<thead>
<tr>
<th>Isotope</th>
<th>$^{102}\text{Pd}$</th>
<th>$^{104}\text{Pd}$</th>
<th>$^{106}\text{Pd}$</th>
<th>$^{108}\text{Pd}$</th>
<th>$^{110}\text{Pd}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
</tr>
<tr>
<td>$c_s$</td>
<td>0.51</td>
<td>0.56</td>
<td>0.60</td>
<td>0.64</td>
<td>0.67</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>71</td>
<td>66</td>
<td>92</td>
<td>74</td>
<td>70</td>
</tr>
</tbody>
</table>

point of Ru isotopic chain is situated at $^{104}_{44}\text{Ru}$. Similar results for Pd isotopic chain are given in Tables 3 and 4.

Our results for control parameters suggest, $^{112}_{46}\text{Pd}$ isotope with $c_s = 0.69$ as a candidate for $SO(6)$ dynamical symmetry limit and $^{102}_{46}\text{Pd}$ with $c_s = 0.51$ proposed to be a typical example with $E(5)$ critical point symmetry. Thus, the critical point of Pd isotopic chain is situated at $^{102}_{46}\text{Pd}$.45
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5.1. Energy surfaces diagrams and phase transitions

In the transition from $U(5)$, i.e., spherical limit to the $SO(6)$, namely $\gamma$-unstable limit, one sees that, the evolution of energy surface goes from a pure $\beta^2$ to a combination of $\beta^2$ and $\beta^4$ that has a deformed minimum.\footnote{At the critical point of this transition, energy surface is a pure $\beta^4$ that is not analytically solvable. This energy surface is approximated with a square well that is analytically solvable. Figure 3 shows, in a schematic way, possible evolution of the potential energy surfaces, i.e., Eq. (15), dependence on the deformation parameter $\beta$. In Figs. 4 and 5, energy surfaces for the chain of Ru and Pd isotopes are plotted as a function of $\beta$. In analyzing it can be said that, $\partial^2 E/\partial \beta^2 = 0$ corresponds approximately to a “very flat energy surface” as happens for the $E(5)$ critical point model. Following this approach,}

![Energy surfaces diagrams](image)

Fig. 3. The evolution of the potential energy surfaces from Eq. (15) in the transition from $U(5)$, i.e., spherical limit to the $SO(6)$, namely $\gamma$-unstable limit.

![Energy surfaces](image)

Fig. 4. Potential energy surfaces as a function of the deformation parameter, $\beta$, from Eq. (15) for different isotopes of Ru.
both $^{104}$Ru and $^{102}$Pd have been found to be close to critical point. The quartic dependence of the energy at the critical point and the predictions for spectra and $B(E2)$ transition rates for the energy surfaces $\beta^4$, $\beta^6$, $\beta^8$ which are called respectively, the $E(5)-\beta^4$, $E(5)-\beta^6$, $E(5)-\beta^8$, have been given in Ref. 48. Some comparison of the work performed in the present paper with the variational method used in literature (see Ref. 49, for example) is in place. The main difference between the two methods is that in Refs. 49 and 50, a trial potential (the Davidson potential) containing a parameter ($\beta_0$) is chosen and subsequently the rate of change of the physical quantity (the rate of change of the energy ratios, $R_L = \frac{E(L)}{E(2)}$) is maximized with respect to this parameter, thus determining the parameter value and the value of the physical quantity (the energy ratios), while in the present paper a method based on Catastrophe Theory in combination with a coherent state analysis has been employed to generate the energy surfaces and the evolution of the potential energy surfaces with respect to $\beta$ are considered. Also, the analysis carried out in Ref. 51 concentrated on the spectroscopic properties of the Ru isotopes and on the application of nuclear supersymmetry for a simultaneous description of these nuclei. Here we have considered the geometrical behavior of $U(5) \leftrightarrow SO(6)$ transitional Hamiltonian.

6. Relation to Landau Theory and Critical Exponents

Landau theory of phase transitions was formulated in the late 1930’s as an attempt to develop a general method of analysis for various types of phase transitions in condensed matter physics (especially in crystals). It relies on two basic conditions, namely on (a) the assumption that, free energy is an analytic function of a quantity called order parameter, and on (b) the fact that, expression for the free energy must obey the symmetries of system. Condition (a) is further
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strengthened by expressing the free energy as a Taylor series in the order parameter. In this section, the Landau theory for phase transitions is shown to be a useful approach also for quantum systems such as the atomic nucleus. A detailed comparison of the two frameworks, i.e., ground state quantum phase transition in IBM and Landau Theory of thermodynamic phase transitions, are performed and a detailed analysis of the critical exponents of ground state quantum phase transition is presented. For fluid systems, as we become close to the critical point, some of the quantities of the system are related to the temperature as $f(T) \approx (T - T_c)^\beta$ for some exponents of $\beta$. The similar behaviors may be seen not as a function of temperature, but as a function of some other quantities of the system $f(x) \approx (x)^\beta$. These exponents, $\beta$, are called critical exponents and naturally defined as $\lim_{x \to 0} \frac{\ln f(x)}{\ln(x)}$. Some basic critical exponents in thermodynamics have been employed to describe the evolution of critical systems near the critical points.\textsuperscript{52,53}

The behavior of $E(A, A', A''; \beta)$ in (16a), near the critical point is determined by the signs of the coefficients $A', A''$. The coefficients $A', A''$ which are functions of $c_s$, are written as functions of the dimensionless quantity, $\hat{c}_s = (c_s - c_{s\text{-critical}})/c_{s\text{-critical}}$ where $c_{s\text{-critical}} = 1/2$. The expansion of coefficient $A'$ around the $\hat{c}_s$ is

$$A' = \left\{ -\frac{1}{8} N(N - 1)g - \frac{1}{4} N\alpha + \frac{1}{4} N(N - 1)gc_d + N\alpha c_d^2 + \frac{6}{5} N\delta + 4N\gamma \right\}$$

$$+ \left[ \frac{1}{4} N(N - 1)(c_d - 1) - \frac{1}{2} N\alpha \right] \hat{c}_s + \left[ -\frac{1}{8} N(N - 1)g - \frac{1}{4} N\alpha \right] \hat{c}_s^2. \quad (18)$$

The coefficient $A'$, which changes signs at $c_{s\text{-critical}} = \frac{1}{2}$, is now written in terms of $\hat{c}_s$, as

$$A' = \left\{ -\frac{1}{8} N(N - 1)g - \frac{1}{4} N\alpha + \frac{1}{4} N(N - 1)gc_d + N\alpha c_d^2 + \frac{6}{5} N\delta + 4N\gamma \right\}$$

$$+ \left[ \frac{1}{4} N(N - 1)(c_d - 1) - \frac{1}{2} N\alpha \right] \hat{c}_s, \quad (19)$$

namely, $A'$ is represented as a series in odd powers of $\hat{c}_s$ so that its sign would change at $\hat{c}_s = 0$. Similarly, the expansion of coefficient $A''$ around the $\hat{c}_s$ is

$$A'' = \left\{ \frac{3}{16} N(N - 1)g + \frac{1}{8} N\alpha - \frac{1}{2} N(N - 1)gc_d + \frac{1}{4} N(N - 1)gc_d^2 \right.$$  

$$- \frac{1}{2} N\alpha c_d^2 + \frac{3}{5} N\delta - 2N\gamma \right\}$$

$$+ \left[ \frac{1}{2} N(N - 1)g \left( \frac{3}{4} - c_d \right) + \frac{1}{4} N\alpha \right] \hat{c}_s$$

$$+ \left[ \frac{3}{16} N(N - 1)g + \frac{1}{8} N\alpha \right] \hat{c}_s^2. \quad (20)$$
The stable systems have $A'' > 0$ on both sides of $c_{s-\text{critical}} = 1/2$. Therefore, the series for $A''$ is represented as one of even powers of $\hat{c}_s$

\[
A'' = \left\{ \frac{3}{16} N(N - 1)g + \frac{1}{8} N\alpha - \frac{1}{2} N(N - 1)gc_d + \frac{1}{4} N(N - 1)gc_d^2 \right. \\
- \frac{1}{2} N\alpha c_d^2 - \frac{3}{5} N\delta - 2N\gamma \left. \right\} + \left[ \frac{3}{16} N(N - 1)g + \frac{1}{8} N\alpha \right] \hat{c}_s^2.
\]

(21)

The condition for stability is that $E(A, A', A''; \beta)$ must be a minimum as a function of $\beta$. From Eq. (16a), this condition may be expressed as

\[
\frac{\partial E}{\partial \beta} = 0 = 2A'\beta + 4A''\beta^3,
\]

(22)

where terms above $\beta^4$ are presumed negligible near $c_{s-\text{critical}} = 1/2$. For $c_s < 1/2$, the only real root is $\beta = 0$. For $c_s > 1/2$, the root $\beta = 0$ correspond to a local maximum, and therefore not to equilibrium. The other two roots are found to be

\[
\langle \beta \rangle = \pm \left( -\frac{a'_1 \hat{c}_s}{2C_2} \right)^{1/2},
\]

(23)

where

\[
a'_1 = \frac{1}{4} N(N - 1)g(c_d - 1) - \frac{1}{2} N\alpha
\]

(24a)

and

\[
C_2 = \frac{3}{16} N(N - 1)g + \frac{1}{8} N\alpha - \frac{1}{2} N(N - 1)gc_d + \frac{1}{4} N(N - 1)gc_d^2 \\
- \frac{1}{2} N\alpha c_d^2 - \frac{3}{5} N\delta - 2N\gamma,
\]

(24b)

where only the first terms in the expansions for $A'$ and $A''$ were used. Consequently, the analysis predicts, the equilibrium order parameter near the critical point should depended on the $\hat{c}_s$ as

\[
\langle \beta \rangle = \text{Cte}(\hat{c}_s)^{1/2},
\]

(25)

which means that the critical exponent for order parameter is 1/2. Note that, this behavior is in perfect agreement with general predictions derived in Ref. 32. On the other hand, a very sensitive probe of the phase transitional behavior is the second derivative of ground-state energy with respect to the control parameters,

\[
C(\lambda_i)_{|_{\lambda_i}} = -\frac{\partial^2}{\partial \lambda^2} \varepsilon_0(\lambda),
\]

(26)

(where all $\lambda_j$'s with $j \neq i$ are kept constant).

In the above discussed thermodynamic analogy, $\varepsilon_0(\lambda)$ is replaced by the equilibrium value of the thermodynamic potential $F_0(P, T)$. In our descriptions, by using
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Eq. (16a), the ground state energies are $A, A - \frac{A'^2}{4\alpha}$ for $\beta = 0$ and $\beta \neq 0$ respectively. From Eq. (26) the specific heats are

$$C^+(\beta_0 = 0) = -\left[\frac{g}{2} N(N - 1) + N\alpha + \frac{\alpha}{2}\right], \quad \text{for } c_s < \frac{1}{2}, \quad (27a)$$

$$C^-(\beta_0 \neq 0) = -\left[\frac{g}{2} N(N - 1) + N\alpha + \frac{\alpha}{2}\right] + \frac{a'^2}{2C_2}, \quad \text{for } c_s > \frac{1}{2}. \quad (27b)$$

These results propose no dependence of $C$ on $\hat{c}_s$ either above or below $c_s$-critical = 1/2, therefore, the values for the specific heat exponents are both zero. Also, this result suggests a discontinuity in the heat capacity in the phase transition point which is in the agreement by Landau’s theory. By means of the Ehrenfest classification\textsuperscript{8,56} of the classical phase transitions, we can determine the order of phase transitions. The Ehrenfest classification was originally elaborated in connection with thermal phase transition. It asserts that the transition is of the $k$th order if the $k$th derivative of the free energy with respect to temperature changes discontinuously at the transitional point. It turned out that such a statement is not always applicable since the derivatives may be singular. The use of the Ehrenfest classification for structural transitions of the ground state relies on the analogy between the free energy as a function of temperature and the ground state energy as a function of an external control parameter. Therefore, Eqs. (26), (27a) and (27b) allows to show that, $U(5) \rightarrow SO(6)$ is of the second-order. In order to identify other critical exponents, according to the Landau theory, by the use of Eq. (16a), the potential energy surface becomes as\textsuperscript{52,53}

$$E(\beta) = A - h\beta + A'\beta^2 + A''\beta^4 + \cdots, \quad (28)$$

where “$h\beta$” represents the contribution of the intensive parameter $h$ for points off the $h = 0$ coexistence curve. The equilibrium equation of state is $(\partial E / \partial \beta)|_{\hat{c}_s, h} = 0$ which cause to (for any small $h$)

$$2(a'_1 \hat{c}_s)\beta + 4(C_2)\beta^3 = h. \quad (29)$$

On the other hand, it reduces to its former representation for $h = 0$. The susceptibility may be found as it introduced in Refs. 52 and 53, namely

$$\chi^{-1}_{c_s} = \left(\frac{\partial h}{\partial \beta}\right)_{\hat{c}_s} = 2a'_1 \hat{c}_s + 12C_2\beta^2. \quad (30)$$

For $c_s < 1/2$, $\beta = 0$ and $\chi^{-1}_{c_s} = 2a'_1 \hat{c}_s$, which gives the critical exponent equal to 1. For $c_s > 1/2$, with $h = 0$, Eq. (29) gives $\beta^2 = -(a'_1 \hat{c}_s)/2C_2$, and therefore $\chi^{-1}_T = 4a'_1 (-\hat{c}_s)$ or the critical exponent equal to 1. Along the critical isotherm, (in the phase transition point) which is obtained by $\hat{c}_s = 0$, and $h = 4C_2\beta^3$, we can specify the critical exponent which in this case is equal to 3.
7. Summary and Conclusion

In this paper, critical behaviors of even–even isotopes of Ru and Pd which evolve from spherical to deformed shapes have been studied. A method based on Catastrophe Theory in combination with a coherent state analysis has been employed to generate the energy surfaces. Catastrophe theory allows one to classify phase transitions and to decide if a nucleus is close to criticality. For each isotopic chain, a general fit is performed in such a way that all parameters but one are kept fixed to describe the whole chain. The best isotopes for the $E(5)$ critical symmetry are proposed. The Landau Theory for phase transitions is shown to be a useful approach also for quantal system such as the atomic nucleus. A detailed comparison of the two frameworks, i.e., ground state quantum phase transition in IBM and Landau Theory of thermodynamic phase transitions, are performed while exploiting the concepts of “specific heat” and “susceptibility”. Some basic critical exponents in thermodynamics have been employed to describe the evolution of systems near the critical points. Finally, the analogy between the critical exponents of ground state quantum phase transition and Landau values for the critical exponents of thermodynamic phase transitions are described.

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