Entanglement of a two-level atom and its spontaneous emission near the edge of a photonic band gap

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Abstract

The entanglement of a two-level atom and its radiation field near the edge of a photonic band gap is studied by using the quantum entropy. Unlike the free space case, there is a steady-state entanglement between the atom and its spontaneous emission field even when the atomic transition frequency lies outside the band gap. Moreover, the degree of entanglement, which is due to the formation of atom–photon bound dressed state, depends on the detuning of the atomic transition frequency from the photonic band edge and can be controlled by a controllable photonic band gap crystal.

1. Introduction

Entanglement plays a crucial role in quantum information and can be used for many practical applications, especially for quantum communication [1], cryptography [2], computation [3] and quantum computation [4]. A system consisting of two subsystems is said to be entangled if its quantum state cannot be described by a product of the quantum states of the two subsystems [5]. It is important for quantum information processing to be able to create entangled states in a controllable way. In recent years, many novel methods have been proposed to generate controllable entangled states [6–9]. Some of them are based on quantum interference effects associated with electromagnetically induced transparency (EIT) [10], including atom–atom, atom–photon and photon–photon entangled states [11–14]. As well, entanglement can exhibit the nature of a nonlocal correlation between quantum systems that have no classical interpretation. However, real quantum systems will unavoidably be influenced by surrounding environments. It is well known that the spontaneous emission and absorption properties depend not only on the energy structure of an atom but also on the nature of the surrounding environment, more specifically, on the density of states (DOS) of the radiation field [15]. So, the entanglement of the atom and its spontaneous emission will be affected by the surrounding environment of the atom. From the point of view of the surrounding environment of atoms, photonic band gap (PBG) structures have been shown to have different DOS compared with a free space vacuum field [16]. PBG structures offer unique ways to tailor light and the propagation of electromagnetic waves and have caused growing interest in recent years because it offers the possibility of controlling and manipulating light within a given frequency range through PBG [17]. In this Letter, we are going to study the entanglement between a two-level atom and its spontaneous emission fields near a photonic band edge by using quantum entropy. The time evolution of atomic quantum entropy and the property of entanglement between the two-level atom and its spontaneous emission fields are shown in detail. We show that, there is a steady-state entanglement between the atom and its spontaneous emission field even when the atomic transition frequency lies outside the band gap. Moreover, we show that by changing the detuning of the atomic transition frequency from the photonic band edge the quantum entropy of the atom can be modified. Accordingly, the entanglement between the atom and photon can be controlled by controllable PBG.

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2. Single-atom entanglement

Consider a single two-level atom with an excited-state [2] and a ground state [1]. The upper level [2] is coupled by radiation modes to the ground state [1] and resonant transition frequency \( \omega_{21} \) is assumed to be near the band edge frequency \( \omega_c \) of a PBG. The Hamiltonian describing the dynamics of this system in the interaction picture and the rotating wave approximation takes the form:

\[
H = \sum_k g_k e^{-i\omega_k t} |2⟩⟨1| \dot{b}_k + \text{H.C.}
\]

(1)

Here, \( \dot{b}_k \) is the annihilation operator for the kth vacuum mode with frequency \( \omega_k \), \( g_k \) is the coupling constant between the kth vacuum mode and the atomic transition from the upper level [2] to the ground level [1], and is assumed to be real. \( \dot{b}_k = \omega_k - \omega_{21} \) is the detuning of the radiation mode frequency \( \omega_k \) from the atomic resonant frequency \( \omega_{21} \). The state vector of the system at time \( t \) may be written as

\[
|ψ(t)⟩ = c_2(t)|2⟩[0⟩ + c_1(t)|1⟩[0⟩ + \sum_k c_k(t)|1⟩[k⟩ + |k⟩],
\]

(2)

where the state vectors \( |1⟩[0⟩ \) and \( |2⟩[0⟩ \) describe the atom in its ground and excited states \( |1⟩ \) and \( |2⟩ \) respectively, with no photons present in any vacuum mode, and the state vector \( |1⟩[1⟩ \) describes the atom in its ground state and a single photon in the kth mode with frequency \( \omega_k \). We assume the atom is initially in the superposition state \( |ψ(0)⟩ = c_2(0)|2⟩ + c_1(0)|1⟩ \), with \( |c_1(0)|^2 + |c_2(0)|^2 = 1 \) and \( c_2(0) = 0 \). We substituted the Hamiltonian (1) into the Schrödinger equation and obtained the following set of equations:

\[
\begin{align*}
\dot{c}_2(t) &= \sum_k g_k e^{-i\omega_k t}c_k(t), \\
\dot{c}_1(t) &= c_1(t), \\
\dot{c}_k(t) &= g_k e^{i\omega_k t}c_2(t).
\end{align*}
\]

(3-5)

By formal time integration of Eq. (5) and eliminating \( c_k(t) \) from Eq. (3) we have:

\[
\dot{c}_2(t) = -\int_0^t G(t - t')c_2(t') dt',
\]

(6)

where \( G(t - t') = \sum_k g_k^2 e^{-i\omega_k(t-t')} \) is the retarded Green function. The resulting Green function depends very strongly on the photon density of states of the relevant photon reservoir. For a broadband reservoir, such as free vacuum with the photon dispersion relation \( \omega_k = c k \), one can use the Wigner–Weisskopf approximation [18] to obtain:

\[
G(t - t') = \frac{\gamma}{2} \delta(t - t'),
\]

(7)

where \( \gamma = \frac{1}{2\pi} \frac{4\omega_0^3}{c^2 \omega_c^2} \). Here, \( d_{21} \) is the magnitude of the atomic dipole moment for the transition \( 2 ⟷ 1 \). Thus, in free space, the Green function is proportional to the delta function. This is because free vacuum is an infinitely broad photon reservoir (flat spectrum) and, therefore, its response should be instantaneous. Applying Eq. (7) in Eq. (6) results to:

\[
|c_2(t)|^2 = e^{-\gamma t}.
\]

(8)

The Wigner–Weisskopf approximation, however, is not valid when the density of electromagnetic modes changes rapidly in the vicinity of the atomic transition frequency \( \omega_{21} \). In this case, we must perform the exact summation \( \sum_k g_k^2 e^{-i\omega_k(t-t')} \). For the purpose of discussion we assume that our two-level atom is embedded in a three-dimensional photonic crystal where the photon dispersion is chosen to be isotropic and satisfies the equation [19]

\[
\cos(kL) = \frac{1}{2} (A + D),
\]

(9)

where

\[
A = \frac{(1 + n)^2}{2n} \cos \left( \frac{2n a + b}{c} \right),
\]

(10)

\[
D = -\frac{(1 - n)^2}{2n} \cos \left( \frac{2n a - b}{c} \right).
\]

(11)

This equation is obtained from the exact solution of the scalar wave equation with dielectric constant \( \epsilon(x) = \sum u(x - mL) \), where

\[
u(x) = \begin{cases} n^2 - 1, & \text{if } |x| < a, \\ 0, & \text{otherwise}. \end{cases}
\]

(12)

The constant \( k \) is the Bloch wave number, \( n \) is the refractive index of the scatterer, \( a \) is its radius, and \( 2a + b = L \) is the lattice constant. Regimes where \( \frac{1}{2}(A + D) < 1 \) correspond to real \( k \) and thus propagating Bloch waves, while \( \frac{1}{2}(A + D) > 1 \), \( k = \pi n / L + i k \), and has an imaginary part \( k \) so that the Bloch wave is evanescent. These are the so-called forbidden bands of periodic medium. The band edges are the regimes where \( \frac{1}{2}(A + D) = 1 \). For \( b = 2na \) Eq. (9) can be solved analytically to give

\[
\omega_k = \frac{c}{4na} \arccos \left( \frac{4n \cos(kL) + (1 - n)^2}{(1 + n)^2} \right).
\]

(13)

By symmetrizing \( k \) to all directions in \( \mathbf{k} \) space, we produce PBG at the spheres \( |\mathbf{k}| = m\pi / L \), with \( m = 1, 2, 3, \ldots \). For \( k \equiv \frac{\pi}{L} \) the photon dispersion relation near the upper band edge \( \omega_k \) can be approximated by [19]:

\[
\omega_k = \omega_c + A(k - k_0)^2,
\]

(14)

with \( A \approx \omega_c / k_0^2 \approx c^2 / \omega_c \) (this corresponds to a density of states of the form \( \rho(\omega) \sim \frac{(\omega - \omega_c)}{\omega_c} \), with \( \theta \) being the Heaviside step function [19]). The dispersion relation (14) is valid for frequencies close to the upper photonic band edge. If the PBG is large, and if the relevant atomic transitions are near the upper band edge, it is a very good approximation to completely neglect the effects of the lower band. With Eq. (14), we can evaluate the Green function as [15]:

\[
G(t - t') = \beta^{3/2} \frac{e^{i(\beta(t-t') + \pi/4)}}{\sqrt{\pi(t-t')}}, \quad t > t',
\]

(15)

where \( \beta^{3/2} = \omega_{21}^{3/2} / \sqrt{6\pi \epsilon_0 \omega_c} \) and \( \delta = \omega_{21} - \omega_c \). In contrast to the free space case Eq. (15) demonstrates that there is a contribution in the current dynamics at time \( t \) from previous states of the system at time \( t' \) following an inverse square root dependence. As we will discuss later on, this is due to the partial localization of the emitted photon in the vicinity of the atom where it can be reabsorbed and thus affect the atom’s evolution again after its initial emission.

We continue by taking the Laplace transform of Eq. (6) to derive the explicit time dependence of the atom’s evolution:

\[
\tilde{c}_2(s) = c_2(0) \left( s + G(s) \right)^{-1}.
\]

(16)

Here, \( G(s) = -i(\beta)^{3/2} / (s - i\delta)^{1/2} \), is the Laplace transform of the Green function. The amplitude \( c_2(t) \) is given by the inverse Laplace
transform $c_2(t) = (1/2\pi i) \int_{-i\infty}^{i\infty} e^{\epsilon t} \tilde{c}_2(s) ds$, where the real number $\epsilon$ is chosen so that $s = \epsilon$ lies to the right of all singularities (poles and branch points) of the function $\tilde{c}_2(s)$. So we can obtain $c_2(t)$ as:

$$c_2(t) = c_2(0) \left(2\alpha_1 x_1 e^{\beta x_1^2 t} + \alpha_1 (x_2 + y_2) e^{\beta y_2^2 t}ight) - 3 \sum_{j=1}^{\infty} \alpha_j y_j \left[1 - \phi\left(\sqrt{\beta x_j^2 t}\right)\right] e^{\beta x_j^2 t},$$

(17)

where $\phi(x)$ is the error function and

$$x_1 = (A_+ e^{\pi i/4})/2,$$

(18)

$$x_2 = (A_+ e^{-\pi i/6} - A_- e^{\pi i/6}) e^{-\pi i/4},$$

(19)

$$x_3 = (A_+ e^{\pi i/6} - A_- e^{-\pi i/6}) e^{3\pi i/4},$$

(20)

$$A_\pm = \left[\frac{1}{2} + \frac{1}{2} (1 + \frac{4}{27} \frac{\delta^2}{\beta^3})\right]^{1/2},$$

(21)

$$y_j = \sqrt{x_j^2 \delta}(j = 1, 2, 3),$$

(22)

$$\alpha_j = \frac{y_j}{(x_j - x_i)}(x_j - x_k) (j \neq i \neq k; j, i, k = 1, 2, 3).$$

(23)

For large $\beta t$, one can ignore the terms of higher order than $\beta t^{-3/2}$, and Eq. (17) reduces to

$$c_2(t) \cong c_2(0) \left(2\alpha_1 x_1 e^{\beta x_1^2 t} + \alpha_1 (x_2 + y_2) e^{\beta y_2^2 t}\right) + c_2(0) \frac{1}{2\sqrt{\pi}} \left(3 \sum_{j=1}^{\infty} \frac{\alpha_j}{x_j^2}\right) e^{\beta x_j^2 t},$$

(24)

The entropy of the atom can be defined through its respective reduced density operator by

$$S_A(t) = -\text{Tr}[\rho_A(t) \ln \rho_A(t)].$$

(25)

Here $\rho_A(t)$ is the reduced density operator of the atom. To obtain the reduced density operator of the atom we rewrite the wave function $|\psi(t)\rangle$ (Eq. (2)) as:

$$|\psi(t)\rangle = |Q\rangle |1\rangle + |P\rangle |1\rangle,$$

(26)

with

$$|Q\rangle = c_2(t) |0\rangle,$$

(27)

$$|P\rangle = c_1(t) |0\rangle + \sum_k c_k(t) |1_k\rangle.$$

(28)

The density operator of the atom-field is given by:

$$\rho_{AF}(t) = |\psi(t)\rangle\langle\psi(t)|,$$

(29)

and the reduced density operator of the atom can be obtained as:

$$\rho_A(t) = \text{Tr}_f\{\rho_{AF}(t)\} = \left[|Q\rangle\langle Q| + |P\rangle\langle P|\right],$$

(30)

where

$$|Q\rangle\langle Q| = c_2(t)^2,$$

(31)

$$|P\rangle\langle P| = c_1(t) c_2(t),$$

(32)

$$|P\rangle\langle P| = c_1(t)^2 + \sum_k c_k(t)^2.$$

(33)

3. Results and discussion

In what follows, we study the entanglement of a two-level atom embedded in a three-dimensional photonic crystal to its spontaneous emission field. We assumed that the atom is initially prepared in the upper level $|2\rangle$ (i.e. $c_1 = 0$ and $c_2 = 1$). So the atom and the radiation-field reservoir are initially in a disentangled pure state. Accordingly the considered system is a bicomponent quantum system in a pure state. For a bicomponent system in a pure state, Bennett et al. [20] and Phoenix et al. [21] have shown that the reduced quantum entropy is a very accurate measure of the degree of the entanglement between two components. The higher the reduced quantum entropy, the greater the entanglement. Some works on the quantum entropy and the entanglement in the bicomponent system was reported [21]. Based on the atomic spontaneous emission systems can be considered as a quantum system with two components, the atom and its spontaneous emission fields. For such a system, we can use the reduced quantum entropy given by Eq. (25) as a measure of the degree of the entanglement between the atom and its spontaneous emission field. To see the effect of PBG on the entanglement, we plotted the evolution of the entanglement as a function of scaled time ($\beta t$) for various values of detuning from the photonic band edge in Fig. 1. Here we considered three different cases of (a) the atomic transition frequency near the band gap, (b) the atomic transition frequency well inside the band gap, and (c) the atomic transition frequency far inside the gap. In order to explain the behavior of the entanglement between the atom and its spontaneous emission field, consider the transition frequency $\omega_{21}$ of the atom to be within the band gap ($\omega_{21} < \omega_c$). If the atom drops to the ground state via spontaneous emission the resulting photon state will be exponentially decaying away from the atom, since the frequency $\omega_k$ of the emitted photon lies within the classically forbidden energy gap of the PBG material. In other words, the spontaneously emitted photon will tunnel through the crystal for a short length, called the localization length, before being Bragg reflected back to the emitting atom to re-excite it. The result is a strongly coupled eigenstate of the electronic degrees of freedom of the atom and the electromagnetic modes of the dielectric. This is the atom–photon bound state first predicted by John and Wang [19]. When the atomic transition frequency is deeply inside the gap the photon tunneling distance is on the scale of few optical wavelengths. As $\omega_{21}$ approaches the band edge $\omega_c$, the photon localization length $\xi_{loc}$ grows larger and eventually diverges near $\omega_c$: $\xi_{loc} \sim c/\sqrt{\omega_c(\omega_c - \omega_{21})}$. In free space, Lamb shift of atomic levels is dominated by the emission and re-absorption of high-energy virtual photons. Within a PBG, this self-dressing is dominated by the real, bound photon. In general, this will lead to some anomalous Lamb shift. If an atomic level lies near a photonic band edge, a more striking effect is predicted to occur [15]. In this case the atom is resonantly coupled to photons of vanishing group velocity. The resultant self-dressing of the atom by its own localized radiation field is sufficiently strong to split the atomic level into a doublet. From Eqs. (17) and (24) we can easily show that $y_2 = x_2$ for $\delta/\beta \gg -3x_2+\sqrt{x_2^2+2x_2^3}/(1+2x_2^3)^{3/2}$. This means that the atomic level splits into dressed states by the strong interaction between the atom and its own radiation. As one can see from Eq. (17), one member of the doublet is pulled into the gap with frequency located at $\omega_0 - \beta |x_2|^2$. The corresponding dressed state is the atom–photon bound dressed state with no-decay. The other dressed state on the contrary is pushed outside the gap with frequency located at $\omega_0 - \beta \text{Im}(x_2^2)$, where the density of photon states is not zero and exhibits resonance fluorescence. In the case where $\omega_{21}$ is far inside the gap we have $y_2 = -x_2$ and the second term in Eqs. (17) and (24) vanishes, i.e., there is no true
atomic level splitting. In the resent case the branch point contribution yields the quasi-dressed state at the band-edge frequency \( \omega_c \). As a result of the interference between the doublets, spontaneous emission decay of the atom displays an oscillatory behavior which manifests itself as the oscillation in the entanglement (solid line in Fig. 1). On the other hand, for the atomic transition frequency deeply inside the band gap, the localization length is very short, accordingly the spontaneous emission very quickly re-excite the atom to the excited level. So the entanglement of the atom and its spontaneous emission field are weakly entangled when the atomic transition frequency \( \omega_c \) lies outside the band gap. This is quite different from the free-space case where the steady-state population on an excited state is always zero, since all of the excited level population eventually decays to the ground level. The steady-state entanglement between the atom and its spontaneous emission field can be easily found by inspection of Eqs. (17), (25) as:

\[
S_s^{-\infty} = |2a_1 x_1|^2 \ln(2a_1 x_1)^{-2} - \ln(1 - |2a_1 x_1|^2) \tag{34}
\]

Using this equation, we plotted the steady-state entanglement of the atom and its spontaneous emission field as a function of the detuning from the band edge in Fig. 3. Here, the steady-state atomic population on the excited state is shown as well (see dashed line). This figure shows that the atom and its radiation field are weakly entangled when the atomic transition frequency is deeply inside the PBG. In this case the main part of the atomic population is at the excited state and only an insignificant part of atomic population is transferred from the quasi-dressed state to the lower level. This means that the spontaneous emission is mainly suppressed by PBG. Consequently, atom–photon entanglement is very weak. By moving the atomic transition frequency near the PBG, the atomic population at the excited state decreases. This
decrease is correspond to the increase of the spontaneous emission and atom–photon entanglement. Eventually, all of the atomic population transfers to the lower level when the atomic transition frequency is very far from the photonic band edge. So, the spontaneous emission reaches to its maximum value and the atom will be disentangled from its radiation field. The dependence of the entanglement to the detuning of the atomic transition frequency from the photonic band edge can be used to control of the entanglement between the atom–photon by using the controllable PBG.

We mentioned that the steady-state entanglement is related to the formation of atom–photon bound dressed state. To see this, we assumed the excited-state population can transfer to another level, too. If the frequency of this transition lies far from the gap, we can use the Wigner–Weisskopf approximation for the spontaneous emission of this transition. To show the influence of transition to this level on the entanglement, we use the following Hamiltonian to describing the dynamics of the Λ-type three level atom:

$$H = \sum_k g_k e^{-ik\ell_1/2} (\hat{b}_k + \sum_q g_q e^{-ik\ell_1/2} |2\rangle \langle 0| \hat{b}_q + H.C.$$  \hspace{1cm} (35)

Here, $\hat{b}_k$ is the annihilation operator for the $k$th vacuum mode with frequency $\omega_k$, $\delta_k = \omega_k - \omega_{20}$ and $g_k$ is the coupling constant between the $k$th vacuum mode and the atomic transition from the upper level |2⟩ to the lower level |0⟩. The state vector of the system at time $t$ may be written as

$$|\psi(t)\rangle = c_2(t)|2\rangle |0\rangle + c_1(t)|1\rangle |0\rangle + c_0(t)|0\rangle |1\rangle + \sum_k c_k(t) |1_k\rangle |0\rangle + \sum_q c_q(t) |0\rangle |1_q\rangle.$$

Using the Schrödinger equation, the equation of motion for the probability amplitudes are given by:

$$i\dot{c}_2(t) = \sum_k g_k e^{-i\delta_k t} c_k(t) + \sum_q g_q e^{-i\delta_q t} c_q(t),$$  \hspace{1cm} (37)

$$c_1(t) = c_1(0),$$  \hspace{1cm} (38)

$$c_0(t) = c_0(0),$$  \hspace{1cm} (39)

$$i\dot{c}_k(t) = g_k e^{i\delta_k} c_2(t),$$  \hspace{1cm} (40)

$$i\dot{c}_q(t) = g_q e^{i\delta_q} c_2(t).$$  \hspace{1cm} (41)

By formal time integration of Eqs. (40), (41) and eliminating $c_k(t)$ and $c_q(t)$ from Eq. (37) we can use the same procedure mentioned above to obtain

$$\dot{c}_2(s) = c_2(0) [s + \tilde{G}(s)]^{-1},$$  \hspace{1cm} (42)

with $\tilde{G}(s) = -((\beta)^{3/2} / (s - i(\delta + \delta_0 - i\gamma_{20}/2))^{1/2}$, where $\delta_0$ and $\gamma_{20}$ are the Lamb shift and spontaneous emission decay of the transition |2⟩ → |0⟩. The inverse Laplace transform of Eq. (42) gives exactly the same form of $c_2(t)$ in Eq. (17) except $\delta$ in Eq. (21) must be replaced by $\delta + \delta_0 - i\gamma_{20}/2$. So, we can use again Eqs. (17), (25) to describe the entanglement between the atom and its spontaneous emission field. To show the effect of transition to the third level on the entanglement, we plotted the entanglement of the atom to its spontaneous emission field as a function of $\beta t$ for different $\gamma_{20}$ with for different $\gamma_{20}$ with $\delta = 0$ in Fig. 4. Because of this new rout, the transition of the upper level population to the lower levels prevents the formation of atom–photon bound dressed state. So, we see that the atom and its spontaneous emission field will be disentangled at the steady-state except for $\gamma_{20} = 0$. Moreover, we see that for $\gamma_{20} \neq 0$ the time of disentanglement decreases with increase of the decay rate.

In our discussion we used isotropic relation Eq. (14) to describe the photon dispersion near the band edge of PBG material. As one know, there is no physical PBG material with an isotropic gap, and a more realistic picture needs to use the anisotropic photon dispersion $\omega_k = \omega_0 + A(k - k_0)^2$. However, the isotropic model exhibits many features of the entanglement in three-dimensional photonic crystals. The most significant difference between the isotropic and anisotropic models is related to the fractionalized steady-state atomic population. For the isotropic model, vacuum Rabi splitting and fractionalized steady-state atomic population occur for all $\omega_{21} \equiv \omega_0$. For the anisotropic model, fractionalized inversion occurs only when $\omega_{21}$ is slightly below $\omega_0$.

4. Conclusion

In summery, we have studied the entanglement of a two-level atom and its spontaneous emission near the edge of a PBG by using the quantum entropy. We discussed the influences of the PBG on the entanglement. The results show that, there is only a transient entanglement between the two-level atom embedded in free space and its spontaneous emission fields. While, our studies indicate that, there is a steady-state entanglement between the atom and its spontaneous emission field even when the atomic transition frequency lies outside the band gap. The degree of entanglement depends on the detuning of the atomic transition frequency from the photonic band edge and can be controlled by the controllable PBG. We demonstrated the steady-state entanglement is related to the formation of atom–photon bound dressed state. In an open system, we showed that due to the lack of atom–photon bound dressed state, the atom will be disentangled from the spontaneous emission field at the steady-state.

References


Fig. 4. The evolution of the entanglement as a function of $\beta t$ for various values of spontaneous emission decay rate $\gamma_{20}$ in the case of $\delta = 0$. 

![Fig. 4](image-url)
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    5238.