Simultaneous description of low-lying positive and negative parity states in \( {\text{spd}}, \ {\text{sdf}} \) and \( {\text{spdf}} \) interacting boson model

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In order to investigate negative parity states, it is necessary to consider negative parity-bosons additionally to the usual \( s \)- and \( d \)-bosons. The dipole and octupole degrees of freedom are essential to describe the observed low-lying collective states with negative parity. An extended interacting boson model (IBM) that describes pairing interactions among \( s, \ p, \ d \) and \( f \)-boson based on affine \( \text{SU}(1,1) \) Lie algebra in the quantum phase transition (QPT) field, such as \( {\text{spd}} \)-IBM, \( {\text{sdf}} \)-IBM and \( {\text{spdf}} \)-IBM, is composed based on algebraic structure. In this paper, a solvable extended transitional Hamiltonian based on affine \( \text{SU}(1,1) \) Lie algebra is proposed to describe low-lying positive and negative parity states between the spherical and deformed gamma-unstable shape. Three model of new algebraic solution for even–even nuclei are introduced. Numerical extraction to low-lying energy levels and transition rates within the control parameters of this evaluated Hamiltonian are presented for various \( N \) values. We reproduced the positive and negative parity states and our calculations suggest that the results of \( {\text{spdf}} \)-IBM are better than \( {\text{spd}} \)-IBM and \( {\text{sdf}} \)-IBM in this literature. By reproducing the experimental results, the method based on signature of the phase transition such as level crossing in the lowest excited states is used to provide a better description of Ru isotopes in this transitional region.

Keywords: Quantum phase transition (QPT); dual algebraic; \( {\text{spd}}, {\text{sdf}} \) and \( {\text{spdf}} \) interacting boson model; positive and negative parity states; \( B(E1) \) and \( B(E2) \) transition rates.
1. Introduction

In recent years, quantum phase transitions (QPTs) as a fundamental and an important issue in a nuclear structure are investigated systematically in even–even nuclei by a variation of parameters in the quantum Hamiltonian. QPTs within the context of the interacting boson model (IBM), correspondence to extensions in nuclear structure, have been studied in a number of recent papers by some group.\(^1\)–\(^4\) Phase transitions associated with a change of shape are known in dynamical systems within the \(U(6)\) symmetry which spanned by combinations of five pairs of \(d\)-boson operators \((L = 2)\) and one pair of \(s\)-boson operators \((L = 0)\), respectively. Based on previous studies, the three possible phases that can occur in the IBM for nuclei have been classified as the spherical vibrator, the axial symmetric rotor and the \(\gamma\)-soft model.\(^5\)–\(^7\) Recently, Iachello has developed a new critical symmetries: \(E(5)\) as a critical point for the transition from spherical to deformed \(\gamma\)-unstable and the \(X(5)\) critical symmetry for a transition between spherical and axially symmetric deformed nuclei.\(^8\)\(^,\)\(^9\) More involved versions of the IBM investigated in several directions like the IBM-2 which no distinction is made between proton pairs and neutron pairs in the framework of an IBM.\(^10\) This kind of analysis has usually been carried out in the IBM-1. The description of odd nuclei can be achieved when valence states of a single fermion are added into the Hilbert space within the IBFM.\(^11\),\(^12\) For several years, great effort has been devoted to the reproduction of negative parity states. If negative parity states are involved within the IBM, besides \(p\)-bosons, \(f\) bosons should also be considered because \(f\)-boson is formed in much lower energy region. Namely, the excitation energy of \(3^-\) state is lower than that of \(1^-\) states, which is common in medium and heavy nuclei. Another extension consists in the introduction of different angular momenta \((p; f; g)\) together with the \(s\) and \(d\) boson like \(spd\)-IBM, \(sdf\)-IBM and \(spdf\)-IBM done by many authors.\(^6\),\(^13\)–\(^18\) In the nuclear model of the IBM, the bosons with other angular momentum are useful mostly as supplements of the dominant dipole and octupole degrees of freedom, which are carried by the system of \(s\)- and \(d\)-boson.\(^19\) The importance of the \(p\)-boson and \(f\)-boson in transitional deformed nuclei has been investigated by Engel\(^7\),\(^20\),\(^21\) who argued the phenomenological studies such as, collective octupole states, low-lying \(E1\) strength, pygmy and giant dipole resonances in IBM. An odd-parity \(p\)- and \(f\)-boson was added to \(sd\)-boson in a framework of the IBM to describe low-lying negative and positive parity states respectively, built on a dipole and octupole excitation. An alternative to the \(spd\) and \(sdf\)-IBM is to use the full \(spdf\) space. For example, the \(spdf\)-IBM model can also be seen as a treatment of \(S, P, D\) and \(F\)-wave pairing in many body structures. When odd parity-bosons are introduced alongside even parity-bosons, there are 16 boson operators \(b_\alpha(\alpha = 1, \ldots, 6)\) \(s(0), p(0, 1), d(0, 1, 2)\) and \(f(0, 1, 2, 3)\), spanning a 16-dimensional (16D) space. The nuclear shape among which the transitions took place is associated with \(SU(5)_{sd}\), \(O(6)_{sd}\), and \(SU(3)_{sd}\) dynamic symmetries of the.
IBM-1 model. The $sd$-boson part of the model is identical to the original version of IBM, which has been successful in describing positive-parity quadrupole collective states. In accordance with the extension from $sd$ space to $spd$, $sdf$ and $spdf$ space, the symmetry group $U(6)_{sd}$ is replaced by $U(9)_{spd}$, $U(13)_{sdf}$ and $U(16)_{spdf}$, respectively. Alternative solvable description of the critical point symmetry in the IBM within framework of $SU(1,1)$ Lie algebra is reported in Ref. 22, but this model is related to the description of positive parity levels. In order to clarify the structural change at the critical point of the phase transition, and investigation of negative parity states the Hamiltonian must be diagonalized numerically similar to what we have done in Refs. 23 and 25. Current research on extension of IBM is focused on the affine $SU(1,1)$ algebraic technique, which determines the properties of nuclei in the $U(5) \rightarrow SO(9)$, $U(5) \rightarrow SO(13)$ and $U(5) \rightarrow SO(16)$ transitional region of IBM.

We started by investigating the $spdf$-IBM, also in $spdf$ some differences compared to the $spd$ and $sdf$ model. One of such differences is found in energy spectra according to an intensity of pairing between $p$, $d$ and $f$-bosons. Another difference has been known in $E1$ transitions. From the calculations, one can see that the good reproduction of the experimentally observed $E1$ transition is dominated by pairing of $d$- and $f$-bosons with no contribution from the $p$-boson ($p$-boson is presumably weakly coupled). We modify the Hamiltonian with regard to describing the negative-parity states. Here, we examine a similar Hamiltonian which is based on $SU(1,1)$ algebraic technique in the $spdf$-IBM to describe the low-lying negative parity states in addition to positive parity states and also the $E1$ transition rates. In particular, the different transitions between spherical and gamma soft nuclei occur in multi-level of IBM, namely $U(5) \rightarrow SO(9)$, $U(5) \rightarrow SO(13)$ and $U(5) \rightarrow SO(16)$.

The remainder of the paper is organized as follows: In Sec. 2, theoretical aspects of transitional Hamiltonian and the affine $SU(1,1)$ algebraic technique will be discussed. Section 3 is related to the investigation of numerical results from applying the considered Hamiltonian to different isotopes. Finally, Sec. 4 summarizes the results of this work and conclusions.

2. Theoretical Framework

The method for diagonalization of Hamiltonian in the transitional region is not as easy as in either of the limits, especially when the dimension of the configuration space is relatively large. To avoid these problems, an algebraic Bethe ansatz method within the framework of an infinite dimensional $SU(1,1)$ Lie algebra has been proposed by Pan et al. In this paper, the method outlined in Ref. 23 will be extended and applied to $spd$, $sdf$ and $spdf$-boson systems which provide spectra of negative and positive parity states in the transitional region. Negative parity levels have been investigated in the particular approaches by involvement of dipole or octupole degrees of freedom. For example, the low-lying positive-parity states of even–even nuclei are shown in Fig. 1, so this model could not explain the negative
The new transitional Hamiltonian is used to investigate the energy spectra and transition probabilities of Ru isotopes, where the extracted control parameters indicate a spherical to deformed shape transition. Also, the special values of control parameters, level crossing and expectation values which are found for Ru nucleus suggest it as the best candidate for the critical point symmetry in this isotopic chain.

2.1. The SU(1, 1) expression of Bethe ansatz equations

In order to analyze the structural changes between the spherical and rotational shapes, similar to what we have done in Refs. 27–29, the Lie algebra corresponding to the SU(1, 1) group is generated by the operators $S^x$ where $x = 0$ and $\pm 1$. The purpose of this approach, particle number-conserving solution to the generalized pairing problem, is derived by introducing an infinite dimensional SU(1, 1) algebra. The structure of dual algebraic including the generators and Casimir operators is presented between the unitary and quasispin. 28 For extension of the model, we introduce the $s$, $p$, $d$ and $f$-bosons SU(1, 1) pairing algebras with

$$S^+(s) = \frac{1}{2}s^\dagger s, \quad S^-(s) = \frac{1}{2}s^2, \quad S^0(s) = \frac{1}{4}(s^\dagger s + s.s^\dagger) = \frac{1}{2}n_s + \frac{1}{4},$$

$$S^+(p) = \frac{1}{2}p^\dagger.p, \quad S^-(p) = \frac{1}{2}p\cdot\tilde{p}, \quad S^0(p) = \frac{1}{4}\sum_{\nu}(p_{\nu}^\dagger p_{\nu} + p_{\nu} p_{\nu}^\dagger) = \frac{1}{2}n_p + \frac{3}{4},$$

$$S^+(d) = \frac{1}{2}d^\dagger d, \quad S^-(d) = \frac{1}{2}d.d, \quad S^0(d) = \frac{1}{4}\sum_{\nu}(d_{\nu}^\dagger d_{\nu} + d_{\nu} d_{\nu}^\dagger) = \frac{1}{2}n_d + \frac{5}{4},$$
Simultaneous description of low-lying positive and negative parity

\[ S^+(f) = \frac{1}{2} f^+ \cdot f^+, \quad S^-(f) = \frac{1}{2} f \cdot f, \quad S^0(f) = \frac{1}{4} \sum_q (f^+_q \cdot f_q + f_q^+ \cdot f^+_q) = \frac{1}{2} n_f + \frac{7}{4}, \]

(4)

where \( n_s, n_p, n_d \) and \( n_f \) are the number operators for \( s, p, f \) and \( d \)-bosons via pair creation operator \( S^+ \), pair annihilation operator \( S^- \) and number-conserving operator \( S^0 \), which satisfy the following commutation relations

\[ [S^0(l), S^\pm(l)] = \pm S^\pm(l), \quad [S^+(l), S^-(l)] = -2S^0(l) \]

(5)

with \( l = 0 \) for \( s \)-boson, \( l = 1 \) for \( p \)-boson, \( l = 2 \) for \( d \)-boson and \( l = 3 \) for \( f \)-boson.

The generated \( b^\dagger_q b^\prime_q \) can be linearly combined into angular momentum tensor operator by

\[ B^\kappa_\mu = (b^\dagger \times b)^\kappa_\mu = \sum_{q=-1}^{l} \sum_{q'=-1}^{l} (\langle q, q' | \kappa \mu \rangle b^\dagger_q b^\prime_{q'} \]

(6)

with \( \kappa = 0, 1, \ldots, 2l \) and \( \mu = \kappa, \kappa - 1, \ldots, -\kappa \) or fixed \( \kappa \), in which \( \delta_{\mu} = (-)^{l-\mu} \delta_{\mu} \), and \( \langle q, q' | \kappa \mu \rangle \) are the Clebsch Gordan (CG) coefficients of the \( O(3) \) generate the \( U(2l + 1) \) algebra.

The Casimir operator of \( SU(1, 1) \) can be expressed as

\[ \hat{C}_2(SU(1, 1)) = S^0(l)(S^0(l) - 1) - S^+(l)S^-(l). \]

(7)

So, \( C_2(SU(1, 1)) \) is related to the Casimir operator of \( O(2l + 1) \) with

\[ \hat{C}_2(SU^p(1, 1)) = \frac{1}{4} \hat{C}_2(O(3)) + \frac{3}{16} \]

(8)

and

\[ \hat{C}_2(SU^d(1, 1)) = \frac{1}{4} \hat{C}_2(O(5)) + \frac{5}{16} \]

(9)

and

\[ \hat{C}_2(SU^f(1, 1)) = \frac{1}{4} \hat{C}_2(O(7)) + \frac{7}{16}. \]

(10)

The representation is determined by a single number \( \kappa \). Let \( |\kappa \mu \rangle \) be a basis vector of an irrep of \( SU(1, 1) \), where \( \kappa \) can be any positive real number, and \( \mu = \kappa, \kappa + 1, \ldots \) Then

\[ \hat{C}_2(SU(1, 1)) |\kappa \mu \rangle = \kappa (\kappa - 1) |\kappa \mu \rangle, \quad S^0(l) |\kappa \mu \rangle = \mu |\kappa \mu \rangle. \]

(11)

Since the basis vectors of \( U(2l + 1) \supset SO(2l + 1) \) are simultaneously the basis vectors of \( SU(1, 1)^l \supset U(1)^l \), with \( l = 0 \) for \( s \)-boson, \( l = 1 \) for \( p \)-boson, \( l = 2 \) for \( d \)-boson and \( l = 3 \) for \( f \)-boson, their complementary relation can be expressed as

\[ |N; n\nu n\Delta LM \rangle = \left[ N, \kappa' = \frac{1}{2} \nu + \frac{1}{4} (2l + 1), \mu' = \frac{1}{2} 2 = \frac{1}{4} (2l + 1), n\Delta LM \right], \]

(12)
where $N,n_l,n_\Delta,L$ and $M$ are quantum numbers of $U(N),U(2l+1),\text{SO}(2l+1),\text{SO}(3)$ and $\text{SO}(2)$ respectively, while $n_\Delta$ is an additional quantum number needed in the $\text{SO}(2l+1) \downarrow \text{SO}(3)$ reduction. Therefore, $\kappa^l$ and $\mu^l$ are quantum numbers of $U(2l+1)$ and $U(1)$, respectively. It is clearly shown that in Eq. (12), the allowed quantum numbers for fixed $\nu$ are $n = \nu, \nu + 2, \nu + 4, \ldots$. The quantum number $\nu$ of $O(2l+1)$ is thus called the seniority number.

The focus of recent branching rules has been on the $U(n) \rightarrow \text{SO}(n)$ by Hamermesh, so branching rules for the irreps of the algebras provide the classification of states for these models. However, the pairing models of multi-level are also characterized by an overlaid $U(n_1 + n_2 + \cdots)$ algebraic structure, described further in Ref. 28.

\begin{equation}
U \left( \sum_{i=1}^{k} n_i \right) \supset \left\{ \text{SO} \left( \sum_{i=1}^{k} n_i \right) \right\} \supset \text{SO}(n_i) \otimes \cdots \text{SO}(n_k) \supset \text{SO}_{i \cdots k}(3). \tag{13}
\end{equation}

The Casimir operator of $\text{SO}(16)$, $\text{SO}(13)$, $\text{SO}(9)$, $\text{SO}(6)$ and that of $\text{SU}^{spd}(1,1)$, $\text{SU}^{adj}(1,1)$, $\text{SU}^{spd}(1,1)$ and $\text{SU}^{adj}(1,1)$ have the following correspondence:

\begin{equation}
\hat{C}_2(\text{SU}^{adj}(1,1)) = \frac{1}{4} \hat{C}_2(\text{SO}(6)) + \frac{3}{4} \tag{14}
\end{equation}

and

\begin{equation}
\hat{C}_2(\text{SU}^{spd}(1,1)) = \frac{1}{4} \hat{C}_2(\text{SO}(9)) + \frac{45}{16} \tag{15}
\end{equation}

and

\begin{equation}
\hat{C}_2(\text{SU}^{adj}(1,1)) = \frac{1}{4} \hat{C}_2(\text{SO}(13)) + \frac{117}{16} \tag{16}
\end{equation}

and

\begin{equation}
\hat{C}_2(\text{SU}^{spd}(1,1)) = \frac{1}{4} \hat{C}_2(\text{SO}(16)) + 12. \tag{17}
\end{equation}

In mathematics, an affine Lie algebra is an infinite dimensional Lie algebra that is constructed in a canonical fashion out of a finite dimensional simple Lie algebra. Affine Lie algebras play an important role in quantum mechanics to find the exact solution for problems as quantum phase transition. Starting from a simple $\text{SU}(1,1)$ Lie algebra, the affine Lie algebra in Eqs. (18)–(23) is obtained by adding extra dimension to the algebra and modifying a commutation in a nontrivial way, which is called a central extension. So we want to show infinite dimensional operators as similarly defined by Pan and Draayer and Jafarizadeh et al. In transition from one limit, to another limit we can show the strength of pairing between bosons by control parameters as $c_s, c_p, c_d$, and $c_f$. So these parameters are control parameters in QPTs. Now, we introduce the operators of infinite dimensional $\text{SU}^{spd}(1,1)$,
Simultaneous description of low-lying positive and negative parity

SU^{spdf}(1,1) and SU^{spd}(1,1) algebra similar to the one defined by Feng et al.\textsuperscript{23}

\[ S_{n}^{\pm}(spdf) = c_{s}^{2n+1}S^{\pm}(s) + c_{p}^{2n+1}S^{\pm}(p) + c_{d}^{2n+1}S^{\pm}(d) + c_{f}^{2n+1}S^{\pm}(f) \]  
(18)

and

\[ S_{n}^{0}(spdf) = c_{s}^{2n}S^{0}(s) + c_{p}^{2n}S^{0}(p) + c_{d}^{2n}S^{0}(d) + c_{f}^{2n}S^{0}(f) \]  
(19)

for infinite dimensional SU^{spd}(1,1) and

\[ S_{n}^{\pm}(spd) = c_{s}^{2n+1}S^{\pm}(s) + c_{p}^{2n+1}S^{\pm}(p) + c_{d}^{2n+1}S^{\pm}(d) \]  
(20)

and

\[ S_{n}^{0}(spd) = c_{s}^{2n}S^{0}(s) + c_{p}^{2n}S^{0}(p) + c_{d}^{2n}S^{0}(d) \]  
(21)

for infinite dimensional SU^{spdf}(1,1) and

\[ S_{n}^{\pm}(spd) = c_{s}^{2n+1}S^{\pm}(s) + c_{p}^{2n+1}S^{\pm}(p) + c_{d}^{2n+1}S^{\pm}(d) \]  
(22)

and

\[ S_{n}^{0}(spd) = c_{s}^{2n}S^{0}(s) + c_{p}^{2n}S^{0}(p) + c_{d}^{2n}S^{0}(d) \]  
(23)

for infinite dimensional SU^{spdf}(1,1) where \( c_{s}, c_{p}, c_{d} \) and \( c_{f} \) are real control parameters, and \( n \) can be taken to be 0, ±1, ±2, . . . .

To evaluate energy spectra and transition rates, let us consider \( |lw\rangle \) as the lowest weight state of SU^{spdf}(1,1) algebra which should satisfy

\[ S^{-}(s)|lw\rangle = 0, \quad S^{-}(p)|lw\rangle = 0, \quad S^{-}(d)|lw\rangle = 0, \quad S^{-}(f)|lw\rangle = 0. \]  
(24)

The lowest weight states, \( |lw\rangle \), are actually a set of basis vectors of the chain

\[ U(2l + 2) \supset U(2l + 1) \supset O(2l + 1) \supset O(3) \supset O(2) \]  
which

\[ |lw\rangle^{spdf} = \left| N, \kappa_{s} = \frac{1}{2} \left( \nu_{s} + \frac{1}{2} \right), \mu_{s} = \frac{1}{2} \left( n_{s} + \frac{1}{2} \right), \kappa_{p} = \frac{1}{2} \left( \nu_{p} + \frac{3}{2} \right), \mu_{p} = \frac{1}{2} \left( n_{p} + \frac{3}{2} \right), \kappa_{d} = \frac{1}{2} \left( \nu_{d} + \frac{5}{2} \right), \mu_{d} = \frac{1}{2} \left( n_{d} + \frac{5}{2} \right), \kappa_{f} = \frac{1}{2} \left( \nu_{f} + \frac{7}{2} \right), \mu_{f} = \frac{1}{2} \left( n_{f} + \frac{7}{2} \right), L, M \rightangle, \]  
(25)

where \( N = \nu + \nu_{f} + \nu_{p} + \nu_{d}, n_{d} = \nu_{s} = \nu_{s} = 0 \) or 1 for spdf-IBM and

\[ |lw\rangle^{spd} = \left| N, \kappa_{s} = \frac{1}{2} \left( \nu_{s} + \frac{1}{2} \right), \mu_{s} = \frac{1}{2} \left( n_{s} + \frac{3}{2} \right), \kappa_{d} = \frac{1}{2} \left( \nu_{d} + \frac{5}{2} \right), \mu_{d} = \frac{1}{2} \left( n_{d} + \frac{5}{2} \right), \kappa_{f} = \frac{1}{2} \left( \nu_{f} + \frac{7}{2} \right), \mu_{f} = \frac{1}{2} \left( n_{f} + \frac{7}{2} \right), L, M \rightangle, \]  
(26)
where $N = \nu + \nu_f + \nu_s, n_d = \nu, n_s = \nu_s = 0$ or 1 for sdf-IBM and

$$|w\rangle^{spd} = |N, \kappa_s = 1/2 (\nu_s + 1/2), \mu_s = 1/2 (n_s + 1/2), \kappa_p = 1/2 (\nu_p + 3/2), \mu_p = 1/2 (n_p + 3/2), \kappa_d = 1/2 (\nu_d + 5/2), \mu_d = 1/2 (n_d + 5/2), L, M\rangle,$$

where $N = \nu + \nu_p + \nu_s, n_d = \nu, n_s = \nu_s = 0$ or 1 for spd-IBM. Hence, we have

$$S_n^0(spdf)|w\rangle = \Lambda_n^{pdf}|w\rangle, \quad S_n^0(sdf)|w\rangle = \Lambda_n^{df}|w\rangle, \quad S_n^0(spd)|w\rangle = \Lambda_n^{spd}|w\rangle,$$

and

$$\Lambda_1^{pdf} = \epsilon_s^2 \frac{1}{2} \left( n_s + \frac{1}{2} \right) + \epsilon_p^2 \frac{1}{2} \left( n_p + \frac{3}{2} \right) + \epsilon_d^2 \frac{1}{2} \left( n_d + \frac{5}{2} \right) + \epsilon_f^2 \frac{1}{2} \left( n_f + \frac{7}{2} \right)$$

and

$$\Lambda_1^{df} = \epsilon_s^2 \frac{1}{2} \left( n_s + \frac{1}{2} \right) + \epsilon_d^2 \frac{1}{2} \left( n_d + \frac{5}{2} \right) + \epsilon_f^2 \frac{1}{2} \left( n_f + \frac{7}{2} \right)$$

and

$$\Lambda_1^{spd} = \epsilon_s^2 \frac{1}{2} \left( n_s + \frac{1}{2} \right) + \epsilon_p^2 \frac{1}{2} \left( n_p + \frac{3}{2} \right) + \epsilon_d^2 \frac{1}{2} \left( n_d + \frac{5}{2} \right).$$

The QPT between spherical and rotational nuclei is mainly driven by the nonzero nondiagonal part of boson pairing operator. Quantum shape phase transition occurs in the full space spd-IBM. In the spd-IBM the $U(5)$, limit is represented by $c_d = 1$ and $c_s = c_p = c_f = c = 0$ and the SO(16) limit happened when $c_s = c_p = c_d = c_f = c = 1$. It should be mentioned that we take ($c_d = 1$) as constant value and phase transition between $U(5)$ and SO(16) limits are described with $c_s = c_p = c_f = c \in [0, 1]$. Hence, phase transition between $U(5)$ and SO(9) limits is described with $c_f = 0, c_s = c_p = c \in [0, 1]$ in spd-IBM, and finally phase transition between $U(5)$ and SO(13) limits is described with $c_p = 0, c_s = c_f = c \in [0, 1]$ in sdf-IBM. Also, the value of $c_d$ is fixed to 1 during the transition region.

By using the generators of SU$^{pdf}(1,1)$, SU$^{df}(1,1)$ and SU$^{spd}(1,1)$ algebra, we can build an extended IBM Hamiltonian for transitional region between two limits as,\textsuperscript{23,25}

$$\hat{H} = g S_0^+ S_0^- + \alpha S_0^0 + \beta \hat{C}_2(\text{SO}_f(7)) + \gamma \hat{C}_2(\text{SO}_d(5)) + \eta (\hat{C}_2(\text{SO}_f(3)))$$

$$+ \xi (\hat{C}_2(\text{SO}_d(3))) + \chi (\hat{C}_2(\text{SO}_p(3))) + \delta (\hat{C}_2(\text{SO}_{pdf}(3))),$$

for spd-IBM and

$$\hat{H} = g S_0^+ S_0^- + \alpha S_0^0 + \beta \hat{C}_2(\text{SO}_f(7)) + \gamma \hat{C}_2(\text{SO}_d(5))$$

$$+ \eta \hat{C}_2(\text{SO}_f(3)) + \xi \hat{C}_2(\text{SO}_d(3)) + \delta (\hat{C}_2(\text{SO}_{df}(3))),$$

for sdf-IBM.
for $sdf$-IBM and

$$
\hat{H} = gS_0^+ S_0^- + \alpha S_1^0 + \beta \hat{C}_2(\text{SO}_d(5)) + \gamma \hat{C}_2(\text{SO}_d(3))
+ \eta(\hat{C}_2(\text{SO}_p(3))) + \delta(\hat{C}_2(\text{SO}_{pd}(3))),
$$

(34)

for $spd$-IBM, where $g, \alpha, \beta, \gamma, \eta, \chi, \xi$ and $\delta$ are real parameters.

To find the nonzero energy eigenstates with $k$-pairs, we exploit a Fourier Laurent expansion of the eigenstates of Hamiltonians which contain dependences on several quantities in terms of unknown $c$-number parameters $x_i (i = 1, 2, \ldots, k)$; so eigenvectors of the Hamiltonian for excitations can be written as

$$
|k; \nu, \nu_p, \nu_f \rangle \equiv \sum_{n_i \in \mathbb{Z}} a_{n_1 n_2 \ldots n_k} x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k} S_{n_1}^+ S_{n_2}^+ \ldots S_{n_k}^+ |w\rangle.
$$

(35)

By using the commutation relations given by Eq. (5), it can be verified that all coefficients $a_{n_1 n_2 \ldots n_k}$ in Eq. (35) can be taken to be 1. The wave functions of Eq. (35) can be expressed by

$$
|k; \nu, \nu_p, \nu_f \rangle \equiv \sum_{n_i \in \mathbb{Z}} a_{n_1 n_2 \ldots n_k} x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k} S_{n_1}^+ S_{n_2}^+ \ldots S_{n_k}^+ |w\rangle.
$$

where $N$ is the normalization constant and

$$
S_{x_i}^+ = \frac{e_s}{1-c_s^2 x_i} S^+(s) + \frac{e_p}{1-c_p^2 x_i} S^+(p) + \frac{e_d}{1-c_d^2 x_i} S^+(d) + \frac{e_f}{1-c_f^2 x_i} S^+(f).
$$

(37)

Bethe ansatz method is the process of diagonalizing depends on a set of dimensionless parameters which denote these parameters by $x_i$, so values of the parameters $x_i$ depend on the pair number $N$. By using Eq. (36) and the commutation relations of Eq. (5) which leads to a set of Bethe ansatz equations (BAE), the $c$-numbers $x_i$’s are determined in Eq. (38).

$$
\frac{\alpha}{x_i} = \frac{g c_s^2 \left( \nu_s + \frac{1}{2} \right)}{1 - c_s^2 x_i} + \frac{g c_p^2 \left( \nu_p + \frac{3}{2} \right)}{1 - c_p^2 x_i} + \frac{g c_d^2 \left( \nu_d + \frac{5}{2} \right)}{1 - c_d^2 x_i} + \frac{g c_f^2 \left( \nu_f + \frac{7}{2} \right)}{1 - c_f^2 x_i}
\quad - \sum_{j \neq i} \frac{2g}{x_i - x_j}.
$$

(38)

This yields the eigenvalues $E^{(k)}$ of the above Hamiltonian in the form

$$
E^{(k)} = h^{(k)} + \alpha \Lambda_1^{spdf} + \beta \nu_f (\nu_f + 5) + \gamma \nu_d (\nu_d + 3) + \eta L_f (L_f + 1)
+ \xi L_d (L_d + 1) + \chi L_p (L_p + 1) + \delta L_{pdf} (L_{pdf} + 1),
$$

(39)

$$
E^{(k)} = h^{(k)} + \alpha \Lambda_1^{spdf} + \beta \nu_f (\nu_f + 5) + \gamma \nu_d (\nu_d + 3) + \eta L_f (L_f + 1)
+ \xi L_d (L_d + 1) + \delta L_{df} (L_{df} + 1),
$$

(40)

1650089-9
M. A. Jafarizadeh, A. J. Majarshin & N. Fouladi

\[ E^{(k)} = h^{(k)} + \alpha \Lambda_{pd}^{(k)} + \beta \nu_{d}(\nu_{d} + 3) + \gamma L_{d}(L_{d} + 1) + \eta L_{p}(L_{p} + 1) + \delta L_{pd}(L_{pd} + 1) \]  

(41)

and the quantum number \((k)\) is related to \(N\) by

\[ N = 2k + \nu_{s} + \nu_{p} + \nu_{d} + \nu_{f} \]

for \(spdf\)-IBM and \(N = 2k + \nu_{s} + \nu_{d} + \nu_{f}\) for \(sdf\)-IBM and \(N = 2k + \nu_{s} + \nu_{p} + \nu_{d}\) for \(spd\)-IBM.

It is worth noting that by omitting the coupling of \(c_{f}\) and \(c_{p}\), all relative relations can be expressed as \(spd\) and \(sdf\) model, respectively. A useful and simple numerical algorithm for solving the BAE (38) and extraction of the constants in comparison with experimental energy spectra of considered nuclei is based on using of Matlab software which will be outlined simultaneously. To determine the roots of the BAE with specified values of \(\nu_{s}, \nu_{p}, \nu_{d}\) and \(\nu_{f}\), we solve Eq. (38) with definite values of \(\alpha\) for \(i = 1\) and then use the function “sym-var” in Matlab to obtain all roots.

We then repeat this procedure with different \(\alpha\) to minimize the root mean square deviation, \(\sigma\), between the calculated energy spectra and experimental counterparts which explore the quality of extraction processes. The deviation is defined by the equality

\[ \sigma = \left( \frac{1}{N_{\text{tot}}} \sum_{i_{\text{tot}}} |E_{\text{exp}}(i) - E_{\text{Cal}}(i)|^2 \right)^{\frac{1}{2}} \text{(keV)}, \]  

(42)

where \(N_{\text{tot}}\) is the number of energy levels where included in the extraction processes.

To determine the roots of the BAE with specified values of \(\nu_{s}, \nu_{p}, \nu_{d}\) and \(\nu_{f}\), we solve Eq. (38). We have extracted the best set of Hamiltonian’s parameters, i.e., \(g, \alpha, \beta, \gamma, \eta, \chi, \xi\) and \(\delta\) via the available experimental data.\(^{31}\) Our criterion is quality factor, \(\sigma\), to determine the better model in fitting process with less error than others. In comparison, we obtain good agreement with the data, the \(spdf\)-IBM providing overall slightly better results.

2.2. (\(E1\)) and (\(E2\)) transition rates

Besides the energy spectra of ground and excited states, calculations of transition rates are then needed. The observables such as electric dipole and quadrupole transition rates, \(B(E1)\) and \(B(E2)\), within the low-lying state, provide important information about the nuclear structure and QPTs. In this section, we discuss the calculation of electric transition strengths for multi-level pairings. The extension of IBM in Ru isotopic chain offers an excellent opportunity to study the behavior of the low-lying \(E1\) and \(E2\) strengths in the transitional region from spherical to deformed \(\gamma\)-unstable nuclei. The reduced electric dipole and quadrupole transition probabilities represent the basic nuclear information complementary of the energies of low-lying levels in nuclides. The low-lying levels of even–even nuclei usually decay by one \(E2\) transition. In the \(sd\)-IBM and \(spdf\)-IBM, the \(E2\) transition operators are
Simultaneous description of low-lying positive and negative parity

defined by\textsuperscript{20,23}

\begin{align}
T_{\mu}^{(E2)}(sd) &= Q_2[s^+ \times \tilde{d} + d^+ \times \tilde{s}]^{(2)}_{\mu} + Q_0[d^+ \times \tilde{d}]^{(2)}_{\mu}, \\
T_{\mu}^{(E2)}(spdf) &= \alpha_2[s^+ \times \tilde{d} + d^+ \times \tilde{s}]^{(2)}_{\mu} + \beta_2[d^+ \times \tilde{d}]^{(2)}_{\mu} + \gamma_2[f^+ \times \tilde{p} + p^+ \times \tilde{f}]^{(2)}_{\mu} + \delta_2[p^+ \times \tilde{p}]^{(2)}_{\mu} + \eta_2[f^+ \times \tilde{f}]^{(2)}_{\mu},
\end{align}

where $Q_0, Q_2, \alpha_2, \beta_2, \gamma_2, \delta_2,$ and $\eta_2$ are coefficients of operators, and $s^+, p^+, d^+$ and $f^+$ are the creation operator of $s, p, d$- and $f$-bosons. The reduced transition probability between $J_i \rightarrow J_f$ is obtained by squaring the corresponding reduced matrix elements of the transition operator.\textsuperscript{32,33}

\begin{equation}
B(E\lambda; J_i \rightarrow J_f) = \frac{|\langle J_f \parallel T^{(E\lambda)} \parallel J_i \rangle|^2}{2J_i + 1},
\end{equation}

where $|\langle J_f \parallel T^{(E\lambda)} \parallel J_i \rangle|$ is a reduced matrix element which is defined by the Wigner–Eckart theorem. To calculate electric transition rates, for example, for the $d$-boson part we have the selection rules, where for $d$-boson $\Delta
u_d = \pm 1, |\Delta L| \leq 2$ and $\Delta\nu_d = 0, \pm 2, |\Delta L| \leq 0, 4$ and we do the same formalism for $s, p$ and $f$-bosons.

For $E1$ transitions, the situation is completely different. Low-lying negative parity states in nuclei in IBA model have been of interest for many years because of the $E1$ transition rates which have been observed in collective dipole and octupole modes\textsuperscript{34–36} and recently in low-lying enhanced $E1$ strength in rare earth nuclei.\textsuperscript{15,16}

To paraphrase $E1$ modes, we should expand our model by adding the $p$ and $f$ degrees of freedom. We shall study such $E1$ transitions in terms of the $spd, spdf$ and $spdf$ interacting boson model by adding odd-parity $f$ ($L = 3$) and $p$ ($L = 1$) which is an extension of (standard) $sd$-IBM. To describe the $E1$ transitions, in the $spd, spdf$ and $spdf$ version of the IBM, the $E1$ operators are obtained by a direct coupling of $sp$- and $pd$-bosons for $U(5)$ to $SO(9), sd$- and $df$-bosons for $U(5)$ to $SO(13)$ and $sp, pd$ and $df$ for $U(5)$ to $SO(16)$\textsuperscript{20,21}

\begin{align}
T_{spd}^{(E1)} &= D_0[s^+ \times \tilde{p} + p^+ \times \tilde{s}]^{(1)}_{\mu} + D_2[p^+ \times \tilde{d} + d^+ \times \tilde{p}]^{(1)}_{\mu}, \\
T_{spdf}^{(E1)} &= e \times [(d^+ \times \tilde{f} + f^+ \times \tilde{d})^{(1)} + \chi_1 O_1 + \chi_1' O_1'],
\end{align}

where $O_1$ and $O_1'$ are the following terms:

\begin{equation}
O_1 = [Q_{sd}^{(2)} \times (s^+ \times \tilde{f} + f^+ \times \tilde{s})^{(3)}_{(1)}],
\end{equation}

\begin{equation}
O_1' = \Sigma \sqrt{(2l + 1)(-1)^{l+1}} \begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & L \end{pmatrix} \times [Q_{sd}^{(2)} \times (d^+ \times \tilde{f} + f^+ \times \tilde{d})^{(1)}_{(1)}],
\end{equation}

and where

\begin{equation}
|Q_{sd} = s^+ \times \tilde{d} + d^+ \times \tilde{s}]^{(2)}_{\mu} + \chi_2[d^+ \times \tilde{d}]^{(2)}_{\mu},
\end{equation}
and $E_1$ operator for $spdf$-IBM

$$
T^{(E_1)}_{spdf} = \chi_{sp}[s^+ \times \tilde{p} + p^+ \times \tilde{s}^{(1)}_{\mu}] + \chi_{pd}[p^+ \times \tilde{d} + d^+ \times \tilde{p}^{(1)}_{\mu}]
$$

$$
+ \chi_{df}[d^+ \times \tilde{f} + f^+ \times \tilde{d}^{(1)}_{\mu}].
$$

To evaluate the $B(E2)$ and $B(E1)$ transition ratios, we have calculated the matrix elements of $T(E2)$ operators for $sd$-IBM and $spdf$-IBM and also $T(E1)$ for $spd$-IBM, $sdf$-IBM and $spdf$-IBM between the considered states which are labeled as our model formalism via selection rules and then, we can extract the constant quantities of Eqs. (43), (44), (46), (47) and (51) in comparison with empirical evidence.

3. Numerical Result

3.1. Energy levels of Ru isotopes

Sometime back, neutron-rich the mass region $A = 98–110$ has been of noticeable attention for nuclear structure studies as it shows many interesting features. Ruthenium isotopes are interesting nuclei which also show the spherical to the deformed phase transition. In the series of Ru isotopes, these nuclei (with several negative parity states and $E(1)$ transition) which will be the main object of this publication, play a crucial role in this context. The nuclear structure of neutron deficient $N < 82$ nuclei differs from the neutron-rich $N > 82$ nuclei. With increasing neutron number from one phase $N = 50$ towards another phase $N = 82$ closed shell, the level structure varies rather slowly and there are some shape phase transitions. Numerical extraction to low-lying energy levels and $E2$ transition probability of even mass nuclei such as Cd, Te, Ba, Xe were studied within the framework of the IBM. The Ru isotopes have been previously investigated in Ref. 46. The appropriate candidates for the second-order phase transition between spherical and $\gamma$-soft shapes are Ru isotopes. When the neutron numbers increase from the magic number of $N = 50$, the quadrupole deformation also increases from small values. The QPTs of even–even these isotopes are considered within the framework of the IBA-1 with the aim of identifying the positive and negative parity states in the low-lying region of spectra. It should be noted that the Hamiltonian which was used in Ref. 23, was constructed by the $sd$-pairing operator as a two-body term.

In this section, we have analyzed the properties of the Ru isotopes in order to investigate the ground and excited state spectra and transition rates related to $spd$ and $sdf$ model and full space $spdf$ model. The best fit which we are considering, guarantees that the parameters are well determined. We have therefore computed the positive and negative parity energy spectra for the transitional region via mentioned Hamiltonians. As the examples of our present technique and suggested results, the energy spectra of the Ru isotopes which are determined within the $spd$, $sdf$ and $spdf$-IBM framework are presented in Figs. 2–8. Unfortunately there are no experimental data available on $1^-$ states except $^{100}$Ru, so other experimental data as
Simultaneous description of low-lying positive and negative parity

Fig. 2. Spectroscopic properties of low-lying positive and negative parity states for $^{98}$Ru, which are determined in terms of the three-level and four-level models, in comparison with the experimental counterparts. Excitation energies of levels are expressed in units of keV. Unfortunately, there are no experimental data available on $^1\!^1\!^1\!^1^−$ states except $^{100}$Ru. The experimental data are taken from Refs. 51–57.

Fig. 3. Similar to Fig. 2 which is presented for $^{100}$Ru nucleus.

Fig. 4. Similar to Fig. 2 which is presented for $^{102}$Ru nucleus.
Fig. 5. Similar to Fig. 2 which is presented for $^{104}\text{Ru}$ nucleus.

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Fig. 6. Similar to Fig. 2 which is presented for $^{106}\text{Ru}$ nucleus.

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Fig. 7. Similar to Fig. 2 which is presented for $^{108}\text{Ru}$ nucleus.

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Simultaneous description of low-lying positive and negative parity

Fig. 8. Similar to Fig. 2 which is presented for $^{110}$Ru nucleus.

$3^-$ or $5^-$ are available, excluding roots that can be obtained from $S_k$ symmetry. Also, Tables 1–3 list the best set of the parameters of transitional Hamiltonian, i.e., $g, \alpha, \beta, \gamma, \eta, \chi, \xi$ and $\delta$, which are extracted by the procedures which are explained in Sec. 2 and the least-square fit (LSF) to the available experimental data$^{51–57}$ for excitation energies and transition probabilities of selected states in this work. The results of spdf-IBM suggest with more high accuracy, i.e., minimum $\sigma$ values, in comparison with experimental data and also the spd and sdf-IBM predictions. In Fig. 9, we have shown the variation of the quality factor, $\sigma$, as a function of the control parameter in multi-level pairing formalism. The results for control parameters, $c = 0.54$ for spdf-IBM, $c = 0.48$ for sdf-IBM and $c = 0.46$ for spd-IBM, propose the $^{104}$Ru isotope as the candidate for critical point symmetry $E(5)$ in this transitional region. The lack of experimental negative parity data forces us to select positive and negative parity states simultaneously, nevertheless, the sd-IBM

![Image](image.png)

Table 1. Parameters of the spdf-IBM Hamiltonian, Eq. (32), which are extracted by leastsquare fitting to the experimental data for different isotopes.

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<th>Nucleus</th>
<th>$^{98}$Ru</th>
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<th>$^{102}$Ru</th>
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Note: $N$ is the boson number and $\alpha, \beta, \gamma, \eta, \xi, \chi, \delta$ and $\sigma$ are the parameters of transitional Hamiltonian for each nuclei. The deviation monitors the quality of the LSF. Control parameter ($c$) is related to transition of $c_s$, $c_p$ and $c_f$ in spdf-IBM.
Table 2. Parameters of the \(sdf\)-IBM Hamiltonian, Eq. (33), which are extracted by least-square fitting to the experimental data for different isotopes.

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<td>130</td>
<td>114</td>
<td>80</td>
<td>59</td>
<td>41</td>
</tr>
<tr>
<td>(\beta)</td>
<td>32</td>
<td>-4.3</td>
<td>1.3</td>
<td>1.5</td>
<td>-6.3</td>
<td>-8.1</td>
<td>-6.5</td>
</tr>
<tr>
<td>(\gamma)</td>
<td>-176</td>
<td>4.2</td>
<td>8.6</td>
<td>1.6</td>
<td>-3.5</td>
<td>5.9</td>
<td>-5.3</td>
</tr>
<tr>
<td>(\eta)</td>
<td>-2.7</td>
<td>15</td>
<td>8.7</td>
<td>-13.3</td>
<td>1.8</td>
<td>1.4</td>
<td>-8</td>
</tr>
<tr>
<td>(\xi)</td>
<td>61.4</td>
<td>-6.1</td>
<td>-3.7</td>
<td>3.9</td>
<td>-1.5</td>
<td>0.002</td>
<td>2.8</td>
</tr>
<tr>
<td>(\delta)</td>
<td>2.8</td>
<td>9.3</td>
<td>20.1</td>
<td>12.5</td>
<td>22.1</td>
<td>13.8</td>
<td>19</td>
</tr>
<tr>
<td>(\sigma)</td>
<td>170</td>
<td>126</td>
<td>180</td>
<td>203</td>
<td>155</td>
<td>162</td>
<td>113</td>
</tr>
</tbody>
</table>

Note: \(N\) is the boson number and \(\alpha, \beta, \gamma, \eta, \xi, \delta\) and \(\sigma\) are the parameters of transitional Hamiltonian for each nuclei. The deviation monitors the quality of the LSF. Control parameter \((c)\) is related to transition of \(c_s\), and \(c_f\) in \(sdf\)-IBM.

Table 3. Parameters of the \(spd\)-IBM Hamiltonian, Eq. (34), which are extracted by least-square fitting to the experimental data for different isotopes.

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>(^{98}\text{Ru})</th>
<th>(^{100}\text{Ru})</th>
<th>(^{102}\text{Ru})</th>
<th>(^{104}\text{Ru})</th>
<th>(^{106}\text{Ru})</th>
<th>(^{108}\text{Ru})</th>
<th>(^{110}\text{Ru})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N)</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td>(c)</td>
<td>0.18</td>
<td>0.34</td>
<td>0.40</td>
<td>0.46</td>
<td>0.63</td>
<td>0.75</td>
<td>0.80</td>
</tr>
<tr>
<td>(\alpha)</td>
<td>269</td>
<td>212</td>
<td>117</td>
<td>116</td>
<td>80</td>
<td>60</td>
<td>43</td>
</tr>
<tr>
<td>(\beta)</td>
<td>0.024</td>
<td>-3.6</td>
<td>3.2</td>
<td>-0.019</td>
<td>0.006</td>
<td>-1.2</td>
<td>0.03</td>
</tr>
<tr>
<td>(\gamma)</td>
<td>-0.023</td>
<td>-0.035</td>
<td>-0.025</td>
<td>-5</td>
<td>-0.013</td>
<td>-1.1</td>
<td>-5.5</td>
</tr>
<tr>
<td>(\eta)</td>
<td>5.3</td>
<td>7.8</td>
<td>10.8</td>
<td>1.2</td>
<td>1.2</td>
<td>0.001</td>
<td>1.6</td>
</tr>
<tr>
<td>(\delta)</td>
<td>-9.5</td>
<td>-6.5</td>
<td>18.7</td>
<td>12.9</td>
<td>23.2</td>
<td>12.9</td>
<td>18.3</td>
</tr>
<tr>
<td>(\sigma)</td>
<td>185</td>
<td>157</td>
<td>198</td>
<td>195</td>
<td>122</td>
<td>156</td>
<td>104</td>
</tr>
</tbody>
</table>

Note: \(N\) is the boson number and \(\alpha, \beta, \gamma, \eta, \delta\) and \(\sigma\) are the parameters of transitional Hamiltonian for each nuclei. The deviation monitors the quality of the LSF. Control parameter \((c)\) is related to transition of \(c_s\), and \(c_p\) in \(spd\)-IBM.

It is enough to investigate positive parity states for comparison. In the description of the positive parity collective states, the negative \(p\)- and \(f\)-boson modes are not more important in this case than many noncollective two-quasiparticle excitations. The main purpose of this paper is to expand interacting even \(sd\)-bosons with two-body interaction to the three-level and four-level models with odd parity-bosons as \(p\)- and \(f\)-bosons. In each model, the positive and negative levels have been fitted simultaneously. Also, our main purpose is not to show in clarification which of the considered models: \(spd\), \(sdf\), or \(spd f\) gives a better description of the positive parity collective states or negative parity states separately. But we have fitted positive and negative parity states simultaneously.

The method and calculation procedure in optimum values of the parameters are similar to the Pan et al. formalism. The method and numerical results are related to two steps. (a) Solve the BAE, (b) Extraction by LSF. To find the BAE for an exactly solvable in our models is only half of the procedure. So, the expansion of (36) along with the energy eigenvalue completely determines the functional
Simultaneous description of low-lying positive and negative parity

Fig. 9. Variation of quality factors, \( \sigma \), is presented as a function of control parameters. The spdf-IBM predictions suggest more accuracy than spd and sdf-IBM.

\[ S^+(x_i) \] and the possible roots \((x_i)\) values. At the beginning of solution, we find the quantum numbers and seniority values of every levels and \( k \)-pair excitation are related to the total number of bosons \( N \) and seniority values by the relation of \((N = 2k + \nu_s + \nu_p + \nu_d + \nu_f)\) for spdf-IBM. If there are \( q \) sets of solutions, an additional quantum number, \( \zeta = 1, 2, \ldots, q \), should be introduced to distinguish them from one another. Thus, wavefunction (36) is denoted as \(|k; \zeta; \nu_s; \nu_p; \nu_d; \nu_f; \nu; n_A; LM\rangle\).

For solving the BAE (38) with definite values of \( c \) and \( \alpha \) for \( i = 1 \) to determine the roots of (BAE) with specified seniority values, similar to procedure which has been done in Refs. 23 and 25, we use a useful and simple numerical algorithm by Matlab software with (syms-var) algorithm. Also, because of the \( S_k \) symmetry with respect to permutations among the \((x_1, x_2, \ldots, x_k)\), we exclude those solutions that can be obtained by such root permutations, keeping only one since the others correspond to the same eigenenergy and wavefunction. In what follows, we use a concrete example to show how to numerically find all roots BAE. We take the spdf-IBM with a \( k \)-pair excitation as an example in \(^{99}\text{Ru}\) for ground state in which the parameters \( \nu_s = 1, \nu_p = 0, \nu_d = 0, \nu_f = 0, \alpha = 225 \) and \( c = 0.25 \). Thus, the BAE becomes

\[
\frac{225}{x_i} = 0.093 \frac{1}{1 - 0.06x_i} + 0.093 \frac{1}{1 - 0.06x_i} + 2.5 \frac{1}{1 - x_i} + 0.218 \frac{1}{1 - 0.06x_i} \sum_{j \neq i} \frac{2}{x_i - x_j} \quad (52)
\]

It can be verified that solutions of \([k + 1]\) can be obtained by using Find Root in Matlab from those for \([k]\). Starting from \([k = 1]\), which can be easily calculated by using (syms-var) algorithm, one can get solutions of \([k = 2]\). Hence, an iterative procedure can be set up for other roots. Excluding roots that can be obtained from \( S_k \) symmetry, all inequivalent roots up to \([k = 4]\) are considered. Although the
exact solution is based on a special numerical example, the conclusions apply to the cases with $\alpha > 0$, and $0 < c < 1$ as well. In all of these cases the roots of BAE are real.

Half of the method for optimizing the set of parameters and detailed discussion of the numerical results in the Hamiltonian includes carrying out a (LSF) in the excitation energies of selected states from empirical available data for Ru isotopic chain. We have repeated these processes with different values of considered quantities to optimize the values. Also, we have considered the extraction process in $spd$, $sdf$ and $spd_f$ models for each nucleus to provide energy spectra with minimum $\sigma$ values as compared to the experimental counterparts. In the case of transition probabilities, by the Wigner–Eckart theorem and the Clebsh–Gordan coefficients and using the selection rules, the theoretical data and coefficients of $E_1$, $E_2$ transition operators are extracted by the similar procedure in Refs. 12 and 23. All the transition values are generally consistent with experimental measurements values.

3.2. $B(E1)$ and $B(E2)$ transition rates

The stable even–even nuclei in Ru isotopic chain exhibit an excellent opportunity for studying behavior of the $B(E1)$ and $B(E2)$ transition rates in transitional region from deformed to spherical nuclei for positive and negative parity states. To determine the $Q_0, Q_2, \alpha_2, \beta_2, \gamma_2, \delta_2$, and $\eta_2$ coefficients, we have followed the procedure in Refs. 12, 23 and 25, i.e., we have treated these parameters as function of total boson number, $N$. The transition rates of positive parity states in nuclei, which commonly classified as $B(E2)$ type, have been investigated in the framework of the $sd$ and $spdf$-IBM model. The $E_1$ transition rates are very sensitive to transition operator. To determine the reduced matrix elements of $B(E1)$ and coefficient of these transitions, we have followed the same procedure of $B(E2)$ as LSF of the experimental transitions. For evaluating $B(E2)$ and $B(E1)$, we consider eigenstates Eqs. (32)–(34), where the normalization factor is obtained as

$$N = \left[ \prod_{n=1}^{k} \sum_{i=p}^{k} \sum_{l=0}^{3} \left( \frac{2c_l^2}{(1 - c_l^2 x_{k+1-n})(1 - c_l^2 x_i)} \right) (k - n + \Lambda(0)(l)) \right]^{-1/2}. \quad (53)$$

Unfortunately, there are very few experimental data available on $B(E1)$ for Ru isotopes except $^{100}$Ru. Four $3_1^-$, six $2_1^-$ and one $5_1^-$ negative odd-$L$ states are found where they are all known experimentally. At the moment, 11 negative parity states of $B(E1)$ transitions are calculated in Ru isotopic chain. However, predictions for $B(E2)$ and $B(E1)$ transition rates are presented for some transitions in Tables 4–8 and 10. To determine the coefficient, we have extracted these quantities from the empirical $B(E2)$ and $B(E1)$ values via Least square technique. The values of coefficient listed in Tables 9 and 11 show that in general there is a good agreement between the calculated $B(E2)$ and $B(E1)$ values with the experimental data. As has been pointed out in experimental data, $^{51-58}$ almost the all $E2$ and $E1$ transition
Simultaneous description of low-lying positive and negative parity

Table 4. Comparison of $B(E2)$ values of $^{98}$Ru low-lying positive level energies with those obtained from the $sd$ and $spdf$-IBM where the experimental data of $^{98}$Ru were taken from Ref. 56 and are presented in Weisskopf units (W.u.).

<table>
<thead>
<tr>
<th>Transition ($J_i \rightarrow J_f$)</th>
<th>Experimental</th>
<th>Theory (sd-IBM)</th>
<th>Theory (spdf-IBM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B(E2): 2^+_1 \rightarrow 0^+_1$</td>
<td>32</td>
<td>13.6</td>
<td>28.26</td>
</tr>
<tr>
<td>$B(E2): 4^+_1 \rightarrow 2^+_1$</td>
<td>12</td>
<td>19.95</td>
<td>13.21</td>
</tr>
<tr>
<td>$B(E2): 2^+_2 \rightarrow 2^+_1$</td>
<td>45</td>
<td>28.78</td>
<td>32.40</td>
</tr>
<tr>
<td>$B(E2): 2^+_2 \rightarrow 0^+_1$</td>
<td>1</td>
<td>14.94</td>
<td>9.41</td>
</tr>
<tr>
<td>$B(E2): 6^+_1 \rightarrow 4^+_1$</td>
<td>12.9</td>
<td>15.22</td>
<td>13.33</td>
</tr>
<tr>
<td>$B(E2): 8^+_1 \rightarrow 6^+_1$</td>
<td>2.5</td>
<td>9.3</td>
<td>7.4</td>
</tr>
</tbody>
</table>

Table 5. Comparison of $B(E2)$ values of $^{100}$Ru low-lying positive level energies with those obtained from the $sd$ and $spdf$-IBM where the experimental data of $^{100}$Ru were taken from Ref. 57.

<table>
<thead>
<tr>
<th>Transition ($J_i \rightarrow J_f$)</th>
<th>Experimental</th>
<th>Theory (sd-IBM)</th>
<th>Theory (spdf-IBM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B(E2): 2^+_1 \rightarrow 0^+_1$</td>
<td>35.6</td>
<td>15.7</td>
<td>17.2</td>
</tr>
<tr>
<td>$B(E2): 0^+_2 \rightarrow 2^+_1$</td>
<td>35</td>
<td>39.72</td>
<td>42.1</td>
</tr>
<tr>
<td>$B(E2): 4^+_1 \rightarrow 2^+_1$</td>
<td>51</td>
<td>7.77</td>
<td>9.43</td>
</tr>
<tr>
<td>$B(E2): 2^+_2 \rightarrow 2^+_1$</td>
<td>30.9</td>
<td>6.64</td>
<td>16.17</td>
</tr>
<tr>
<td>$B(E2): 2^+_2 \rightarrow 0^+_1$</td>
<td>1.9</td>
<td>5.3</td>
<td>1.95</td>
</tr>
<tr>
<td>$B(E2): 2^+_2 \rightarrow 4^+_1$</td>
<td>13</td>
<td>29.14</td>
<td>3.5</td>
</tr>
<tr>
<td>$B(E2): 2^+_2 \rightarrow 0^+_1$</td>
<td>37</td>
<td>35</td>
<td>36</td>
</tr>
<tr>
<td>$B(E2): 2^+_2 \rightarrow 0^+_1$</td>
<td>0.44</td>
<td>6.07</td>
<td>0</td>
</tr>
<tr>
<td>$B(E2): 3^+_1 \rightarrow 2^+_1$</td>
<td>10</td>
<td>8.73</td>
<td>5.8</td>
</tr>
<tr>
<td>$B(E2): 3^+_1 \rightarrow 4^+_1$</td>
<td>13.1</td>
<td>15.71</td>
<td>10</td>
</tr>
<tr>
<td>$B(E2): 3^+_1 \rightarrow 2^+_1$</td>
<td>4</td>
<td>11.77</td>
<td>9.9</td>
</tr>
<tr>
<td>$B(E2): 2^+_3 \rightarrow 4^+_1$</td>
<td>27</td>
<td>17.78</td>
<td>30.4</td>
</tr>
<tr>
<td>$B(E2): 2^+_3 \rightarrow 2^+_1$</td>
<td>1.9</td>
<td>13.22</td>
<td>18</td>
</tr>
</tbody>
</table>

Table 6. Comparison of $B(E2)$ values of $^{102}$Ru low-lying positive level energies with those obtained from the $sd$ and $spdf$-IBM where the experimental data of $^{102}$Ru were taken from Ref. 51.

<table>
<thead>
<tr>
<th>Transition ($J_i \rightarrow J_f$)</th>
<th>Experimental</th>
<th>Theory (sd-IBM)</th>
<th>Theory (spdf-IBM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B(E2): 2^+_1 \rightarrow 0^+_1$</td>
<td>45.1</td>
<td>30</td>
<td>49.1</td>
</tr>
<tr>
<td>$B(E2): 0^+_2 \rightarrow 2^+_1$</td>
<td>35</td>
<td>62.3</td>
<td>55.20</td>
</tr>
<tr>
<td>$B(E2): 2^+_2 \rightarrow 2^+_1$</td>
<td>32</td>
<td>35</td>
<td>22.16</td>
</tr>
<tr>
<td>$B(E2): 2^+_2 \rightarrow 0^+_1$</td>
<td>1.14</td>
<td>0</td>
<td>0.94</td>
</tr>
<tr>
<td>$B(E2): 4^+_1 \rightarrow 2^+_1$</td>
<td>66</td>
<td>23.6</td>
<td>66.13</td>
</tr>
<tr>
<td>$B(E2): 3^+_1 \rightarrow 4^+_1$</td>
<td>68</td>
<td>71.80</td>
<td>30.56</td>
</tr>
<tr>
<td>$B(E2): 7^+_1 \rightarrow 5^+_2$</td>
<td>56</td>
<td>61.60</td>
<td>58.18</td>
</tr>
<tr>
<td>$B(E2): 10^+_1 \rightarrow 8^+_1$</td>
<td>57</td>
<td>39.4</td>
<td>60</td>
</tr>
</tbody>
</table>
M. A. Jafarizadeh, A. J. Majarshin & N. Fouladi

3.3. Level crossing

Band crossing in the IBM has been extensively studied mainly in Refs. 59–64. In this section, we have studied the energy level of even–even Ru isotopes up to moderately high spins. It is revealed that the energy of all levels decrease smoothly as a function of control parameters and reaches a minimum at \( N = 62 \) or 64 (\(^{110}\)Ru). Actually, there are several level crossing points, such as those with \( q_1 = 0.45 \) up to 0.65 etc. as shown in Figs. 10–13 for multi-level pairing in transition region of \(^{98–110}\)Ru.

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>( Q_0 )</th>
<th>( Q_2 )</th>
<th>( \alpha_2 )</th>
<th>( \beta_2 )</th>
<th>( \gamma_2 )</th>
<th>( \delta_2 )</th>
<th>( \eta_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(^{98})Ru</td>
<td>–0.43</td>
<td>0.59</td>
<td>3.96</td>
<td>–0.58</td>
<td>–348.07</td>
<td>–465.48</td>
<td>8.02</td>
</tr>
<tr>
<td>(^{100})Ru</td>
<td>–0.37</td>
<td>–0.02</td>
<td>210.15</td>
<td>59.44</td>
<td>393.01</td>
<td>–7.69</td>
<td>7.29</td>
</tr>
<tr>
<td>(^{102})Ru</td>
<td>–3.56</td>
<td>1.62</td>
<td>24.61</td>
<td>–5.49</td>
<td>–268.80</td>
<td>72.97</td>
<td>285.38</td>
</tr>
<tr>
<td>(^{104})Ru</td>
<td>0.159</td>
<td>0.75</td>
<td>–2.7</td>
<td>–185.31</td>
<td>2.11</td>
<td>–6.17</td>
<td>470.72</td>
</tr>
<tr>
<td>(^{108})Ru</td>
<td>6.55</td>
<td>–2.70</td>
<td>–0.56</td>
<td>39.44</td>
<td>161.42</td>
<td>–197.05</td>
<td>–56.51</td>
</tr>
</tbody>
</table>

Table 7. Comparison of \( B(E2) \) values of \(^{104}\)Ru low-lying positive level energies with those obtained from the \( sd \) and \( spdf \)-IBM where the experimental data of \(^{104}\)Ru were taken from Ref. 52.

<table>
<thead>
<tr>
<th>Transition ((J_i \rightarrow J_f))</th>
<th>Experimental</th>
<th>Theory ((sd)-IBM)</th>
<th>Theory ((spdf)-IBM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(B(E2) : 2^+_1 \rightarrow 0^+_1)</td>
<td>57.9</td>
<td>1.17</td>
<td>32</td>
</tr>
<tr>
<td>(B(E2) : 4^+_1 \rightarrow 2^+_1)</td>
<td>83</td>
<td>21</td>
<td>21.94</td>
</tr>
<tr>
<td>(B(E2) : 2^+_2 \rightarrow 2^+_1)</td>
<td>0.6</td>
<td>3.6</td>
<td>3.9</td>
</tr>
<tr>
<td>(B(E2) : 2^+_2 \rightarrow 0^+_1)</td>
<td>2.8</td>
<td>1.3</td>
<td>2.5</td>
</tr>
<tr>
<td>(B(E2) : 0^+_1 \rightarrow 2^+_1)</td>
<td>25</td>
<td>26</td>
<td>12</td>
</tr>
</tbody>
</table>

Table 8. Comparison of \( B(E2) \) values of \(^{108}\)Ru low-lying positive level energies with those obtained from the \( sd \) and \( spdf \)-IBM where the experimental data of \(^{108}\)Ru were taken from Ref. 54.

<table>
<thead>
<tr>
<th>Transition ((J_i \rightarrow J_f))</th>
<th>Experimental</th>
<th>Theory ((sd)-IBM)</th>
<th>Theory ((spdf)-IBM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(B(E2) : 2^+_1 \rightarrow 0^+_1)</td>
<td>58</td>
<td>25</td>
<td>41.72</td>
</tr>
<tr>
<td>(B(E2) : 4^+_1 \rightarrow 2^+_1)</td>
<td>102</td>
<td>101.5</td>
<td>102.04</td>
</tr>
<tr>
<td>(B(E2) : 2^+_2 \rightarrow 0^+_1)</td>
<td>0.5</td>
<td>6.58</td>
<td>1.13</td>
</tr>
<tr>
<td>(B(E2) : 2^+_2 \rightarrow 4^+_1)</td>
<td>0.08</td>
<td>3.18</td>
<td>9</td>
</tr>
<tr>
<td>(B(E2) : 2^+_2 \rightarrow 0^+_1)</td>
<td>0.004</td>
<td>10.24</td>
<td>5.09</td>
</tr>
</tbody>
</table>

Table 9. The effective \( B(E2) \) coefficients of considered isotope which are extracted from experimental data\(^{51–57}\) by Eqs. (43) and (44) in \(sd\)-IBM and \(spdf\)-IBM framework.

Table 10 for \( B(E1) \). Moreover, a nice description of the \( E2 \) and \( E1 \) transition rates of some octupole band members was obtained by other models as \( spd \) and \( sd\)-IBM-1.

Rates of Ru isotopes are quite well reproduced by the \( spdf \)-IBM calculations than \( spd \) and \( sd\)-IBM due to full space dynamical symmetry in Tables 4–8 for \( B(E2) \) and Table 10 for \( B(E1) \).
Fig. 10. Several experimental excited energy levels with variation of the control parameter.

Table 10. Comparison of $B(E1)$ values of $^{100}$Ru low-lying negative level energies with those obtained from the $sp$, $sd$ and $spdf$-IBM where the experimental data of $^{100}$Ru were taken from Ref. 57.

<table>
<thead>
<tr>
<th>Transition ($J_i \rightarrow J_f$)</th>
<th>Experimental</th>
<th>Theory ($sp$-IBM)</th>
<th>Theory ($sd$-IBM)</th>
<th>Theory ($spdf$-IBM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B(E1) : 3_1^- \rightarrow 2_1^+$</td>
<td>$4.0 \times 10^{-5}$</td>
<td>$2.5 \times 10^{-5}$</td>
<td>$4.4 \times 10^{-5}$</td>
<td>$2.5 \times 10^{-5}$</td>
</tr>
<tr>
<td>$B(E1) : 3_1^- \rightarrow 2_2^+$</td>
<td>$1.5 \times 10^{-7}$</td>
<td>$4.6 \times 10^{-7}$</td>
<td>$1.1 \times 10^{-5}$</td>
<td>$6.7 \times 10^{-7}$</td>
</tr>
<tr>
<td>$B(E1) : 3_1^- \rightarrow 2_3^+$</td>
<td>$2.0 \times 10^{-6}$</td>
<td>$2.5 \times 10^{-5}$</td>
<td>$3.6 \times 10^{-7}$</td>
<td>$4.1 \times 10^{-5}$</td>
</tr>
<tr>
<td>$B(E1) : 3_1^- \rightarrow 4_1^+$</td>
<td>$2.3 \times 10^{-7}$</td>
<td>$8.6 \times 10^{-6}$</td>
<td>$1.1 \times 10^{-4}$</td>
<td>$1.5 \times 10^{-5}$</td>
</tr>
<tr>
<td>$B(E1) : 2_1^- \rightarrow 2_2^+$</td>
<td>$27 \times 10^{-5}$</td>
<td>$35 \times 10^{-5}$</td>
<td>$19 \times 10^{-5}$</td>
<td>$43 \times 10^{-5}$</td>
</tr>
<tr>
<td>$B(E1) : 2_1^- \rightarrow 2_3^+$</td>
<td>$33 \times 10^{-5}$</td>
<td>$26 \times 10^{-5}$</td>
<td>$27 \times 10^{-5}$</td>
<td>$37 \times 10^{-5}$</td>
</tr>
<tr>
<td>$B(E1) : 2_1^- \rightarrow 3_1^+$</td>
<td>$53 \times 10^{-5}$</td>
<td>$43 \times 10^{-5}$</td>
<td>$24 \times 10^{-5}$</td>
<td>$27 \times 10^{-5}$</td>
</tr>
<tr>
<td>$B(E1) : 2_1^- \rightarrow 2_4^+$</td>
<td>$2.3 \times 10^{-5}$</td>
<td>$2.1 \times 10^{-4}$</td>
<td>$1.2 \times 10^{-4}$</td>
<td>$2.8 \times 10^{-5}$</td>
</tr>
<tr>
<td>$B(E1) : 2_1^- \rightarrow 2_5^+$</td>
<td>$26 \times 10^{-5}$</td>
<td>$19 \times 10^{-5}$</td>
<td>$68 \times 10^{-5}$</td>
<td>$29 \times 10^{-5}$</td>
</tr>
<tr>
<td>$B(E1) : 2_1^- \rightarrow 4_1^+$</td>
<td>$2.2 \times 10^{-5}$</td>
<td>$9.5 \times 10^{-5}$</td>
<td>$3.2 \times 10^{-5}$</td>
<td>$2.7 \times 10^{-5}$</td>
</tr>
<tr>
<td>$B(E1) : 5_1^- \rightarrow 4_1^+$</td>
<td>$19 \times 10^{-5}$</td>
<td>$16 \times 10^{-5}$</td>
<td>$12 \times 10^{-5}$</td>
<td>$16 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

Table 11. The effective $B(E1)$ coefficients of considered isotope which are extracted from experimental data by Eqs. (46), (47) and (51) in $sp$-IBM, $sd$-IBM and $spdf$-IBM framework.

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>$D_0$</th>
<th>$D_2$</th>
<th>$e$</th>
<th>$\chi_1$</th>
<th>$\chi'_1$</th>
<th>$\chi_{sp}$</th>
<th>$\chi_{pd}$</th>
<th>$\chi_{df}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{100}$Ru</td>
<td>$-0.0214$</td>
<td>$0.0007$</td>
<td>$0.0103$</td>
<td>$0.00011$</td>
<td>$-0.00033$</td>
<td>$-0.0514$</td>
<td>$-0.0002$</td>
<td>$0.0004$</td>
</tr>
</tbody>
</table>
The level crossing point in our model also coincides with the critical point around $c_l \approx 0.50$. It means that the levels touch each other according to levels with different values of angular momentum ($l$) and levels with different seniority $\nu$ because the excited energy levels become degenerate due to having the same roots of BAE. Also, the apparent level crossing suggests a structural change in this transitional...
Simultaneous description of low-lying positive and negative parity

Fig. 13. Several excited energy levels with specific seniority number which cross with each other at the critical-like points with variation of the control parameter for spdf-IBM.

region. We conclude that, in this case, the phase transition with respect to values of the order parameters in the two phases is well defined in the framework of boson system and is due to the level crossing according to the above discussion.

The QPTs are associated with a sudden change in nuclear collective behavior in some quantities such as \( R_{4/2} = E_{4^+_1}/E_{2^+_1} \), i.e., rotational excitation and \( R_{0/2} = E_{0^+_2}/E_{2^+_1} \), i.e., vibrational excitation. In the case of \( R_{4/2} \) the behavior of the chain of isotopes is essential. This quantity is expected to be 2.0 for the vibrator limit and it can increase up to 2.50 which is suggested for the deformed \( \gamma \)-soft limit. Iachello proposed 2.20 for the \( E(5) \) dynamical symmetry characterizing the critical point of spherical to \( \gamma \)-soft transitional region.\(^8\,^9\) Table 12 shows the estimated control parameters and the ratio of \( R_{4/2} \) and \( R_{0/2} \) for the isotopic chain and suggests some nonzero values for the control parameter, namely the ratio of the shape phase

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>( c(\text{spdf}) )</th>
<th>( c(\text{sdf}) )</th>
<th>( c(\text{spd}) )</th>
<th>( R_{4/2} = E_{4^+<em>1}/E</em>{2^+_1} )</th>
<th>( R_{0/2} = E_{0^+<em>2}/E</em>{2^+_1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(^{98}\text{Ru})</td>
<td>0.25</td>
<td>0.24</td>
<td>0.18</td>
<td>2.14</td>
<td>2.02</td>
</tr>
<tr>
<td>(^{100}\text{Ru})</td>
<td>0.36</td>
<td>0.37</td>
<td>0.34</td>
<td>2.27</td>
<td>2.09</td>
</tr>
<tr>
<td>(^{102}\text{Ru})</td>
<td>0.45</td>
<td>0.42</td>
<td>0.40</td>
<td>2.32</td>
<td>1.98</td>
</tr>
<tr>
<td>(^{104}\text{Ru})</td>
<td>0.54</td>
<td>0.48</td>
<td>0.46</td>
<td>2.48</td>
<td>2.75</td>
</tr>
<tr>
<td>(^{106}\text{Ru})</td>
<td>0.65</td>
<td>0.68</td>
<td>0.63</td>
<td>2.64</td>
<td>3.66</td>
</tr>
<tr>
<td>(^{108}\text{Ru})</td>
<td>0.71</td>
<td>0.74</td>
<td>0.75</td>
<td>2.74</td>
<td>4.02</td>
</tr>
<tr>
<td>(^{110}\text{Ru})</td>
<td>0.76</td>
<td>0.79</td>
<td>0.80</td>
<td>2.39</td>
<td>4.73</td>
</tr>
</tbody>
</table>

Note: The \(^{104}\text{Ru}\) isotope with control parameters \( c = 0.54 \) for \( \text{spdf-IBM} \), \( c = 0.48 \) for \( \text{sdf-IBM} \) and \( c = 0.46 \) for \( \text{spd-IBM} \) is related to \( E(5) \) dynamical symmetry.
transition, and also rises when the deformation is increased. Our results for control parameters suggest $^{98}$Ru isotope with control parameters of $c = 0.25$ for spdf-IBM, $c = 0.24$ for sdf-IBM and $c = 0.18$ for spd-IBM as a candidate for the spherical vibrator limit and $^{110}$Ru isotope with control parameters of $c = 0.76$ for spdf-IBM, $c = 0.79$ for sdf-IBM and $c = 0.80$ for spd-IBM as a candidate for the gamma soft dynamical symmetry limit and $^{104}$Ru with $c = 0.54$ for spdf-IBM, $c = 0.48$ for sdf-IBM and $c = 0.46$ for spd-IBM as a typical example for $E(5)$ critical point symmetry.

4. Summary and Conclusion

In this paper, $^{98–110}$Ru isotopic chain was studied in the spherical-$\gamma$ soft rotation transitional region within the spd and sdf-IBM and the full space spdf-IBM framework. A solvable extended transitional Hamiltonian which is based on SU(1, 1) algebra is proposed to provide an investigation of quantum phase transition between the spherical and rotational nuclei and also simultaneous description of low-lying positive and negative parity states in even–even nuclei. This new interaction has been employed to calculate low-lying positive and negative parity energy spectra below 4 MeV of $Z = 44$ isotopes with $N$ between 54 and 66. The agreement with experiment is rather good especially for $B(E1)$ and $B(E2)$ transition probabilities within the framework of multi-level pairing. Our calculations suggest that the results of spdf-IBM are compatible with Ru isotopic chains data than spd and sdf-IBM. The results show that three and four-level pairing many-body models within the extended Hamiltonian describe the structure of nuclei around mass $A \sim 100–110$, especially when the negative parity excited states are taken into account. However, our results show the theoretical calculations are consistent with the experimental data for the considered nuclei.

References

Simultaneous description of low-lying positive and negative parity

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