SPIN–MOMENTUM CORRELATION IN RELATIVISTIC SINGLE-PARTICLE QUANTUM STATES

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This paper is concerned with the spin–momentum correlation in single-particle quantum states, which is described by the mixed states under Lorentz transformations. For convenience, instead of using the superposition of momenta we use only two momentum eigenstates ($p_1$ and $p_2$) that are perpendicular to the Lorentz boost direction. Consequently, in 2D momentum subspace we show that the entanglement of spin and momentum in the moving frame depends on the angle between them. Therefore, when spin and momentum are perpendicular the measure of entanglement is not an observer-dependent quantity in the inertial frame. Likewise, we have calculated the measure of entanglement (by using the concurrence) and have shown that entanglement decreases with respect to the increase in observer velocity. Finally, we argue that Wigner rotation is induced by Lorentz transformations and can be realized as a controlling operator.

Keywords: Spin–momentum correlation; relativistic entanglement; quantum gate.

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1. Introduction

In the last two decades, quantum entanglement has been one of the most important resources in the rapidly growing field of quantum information processing, with remarkable applications on it, and it is based on the fact that the existence of entangled states produces nonclassical phenomena. Therefore, specifying that a particular quantum state is entangled or separable is important because if the quantum state is separable then its statistical properties can be explained entirely by classical statistics.
Relativistic aspects of quantum mechanics have recently attracted much attention in the context of the theory of quantum information, especially on quantum entanglement.\textsuperscript{2–18} Peres et al.\textsuperscript{6} have recently observed that the reduced spin density matrix of a single spin$\frac{1}{2}$ particle is not a relativistic invariant, and Wigner rotations correlate spin with the particle momentum distribution when it is observed in a moving frame.\textsuperscript{7} Gingrich and Adami have shown that the entanglement between the spins of two particles is carried over to the entanglement between the momenta of the particles by the Wigner rotation, even though the entanglement of the entire system is Lorentz-invariant.\textsuperscript{8} Terashimo and Ueda\textsuperscript{9} and Czachor\textsuperscript{10} suggested that the degree of violation of the Bell inequality depends on the velocity of the pair of spin$\frac{1}{2}$ particles or the observer with respect to the laboratory. Alsing and Milburn studied the Lorentz invariance of entanglement and showed that the entanglement fidelity of the bipartite state is preserved explicitly. Instead of a state vector in the Hilbert space, they used a four-component Dirac spinor or a polarization vector in favor of quantum field theory.\textsuperscript{11} Ahn also calculated the degree of violation of the Bell inequality, which decreases with increasing velocity of the observer.\textsuperscript{12} Most of the previous works were concerned with the pure states, although the authors of Refs. 13–15 have considered mixed quantum states that are described by superposition of momenta with Gaussian distribution, where Lorentz transformation introduces a transfer of entanglement between different degrees of freedom; while the entanglement between spins and momenta of particles may change, separately. However, the total entanglement of particle–particle is the same in all inertial frames. Besides the previous works concerned with study of the entanglement between quantum states of two particles, here we generalize this to the spin–momentum correlation of relativistic single particles (by using the concurrence) and show that the measure of entanglement depends on the angle between spin and momentum, and that it decreases with increasing velocity of the observer. Also, it has been shown that the Wigner angle depends on momentum, and so Wigner rotation behaves as a quantum gate or controlling operators. Thus, using this quantum gate the spin–momentum entanglement changes in the framework of special relativity.

This paper is organized as follows. Section 2 is devoted to single-particle relativistic quantum states. In Sec. 3, we calculate explicitly the spin–momentum entanglement of the relativistic quantum state. In Sec. 4, we explain how we can use the quantum gate via Lorentz transformation. The last section contains concluding remarks. The paper also has two appendices.

2. Single-Particle Relativistic Quantum States

Suppose that we have a bipartite system with its quantum degrees of freedom distributed among two parties $\mathcal{A}$ and $\mathcal{B}$ with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively (the standard Hilbert space of dimension $d$ endowed with the usual inner product denoted by $\langle \cdot , \cdot \rangle$). In this paper the quantum state is made up of a single particle having
two types of degrees of freedom: momentum $p$ and spin $\sigma$. The former is a continuous
variable with Hilbert space of infinite dimension but we restrict ourselves here to 2D
momentum subspace with two eigenstates $p_1$ and $p_2$, while the latter is a discrete one
with Hilbert space of spin particles. The pure quantum state of such a system can
always be written as

$$|\psi\rangle = \sum_{i=1}^{2} \sum_{j=-n}^{n} c_{ij} |p_i\rangle \otimes |j\rangle,$$

(2.1)

where $|p_1\rangle$ and $|p_2\rangle$ are two momentum eigenstates of each particle and the kets $|j\rangle$ are the
eigenstates of the spin operator. $c_{ij}$’s are complex coefficients such that $\sum_{ij} |c_{ij}|^2 = 1$.

A bipartite quantum mixed state is defined as a convex combination of bipartite
pure states (2.1), i.e.

$$\rho = \sum_{i=1}^{4} P_i |\psi_i\rangle \langle \psi_i|,$$

(2.2)

where $P_i \geq 0$, $\sum_i P_i = 1$, $|\psi_i\rangle (i = 1, 2, 3, 4)$ as four orthogonal maximal entangled Bell
states (BD) belong to the product space $\mathcal{H}_A \otimes \mathcal{H}_B$ and in terms of momentum and
spin states are well known as

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} (|p_1\rangle \otimes |n\rangle + |p_2\rangle \otimes |-n\rangle),$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} (|p_1\rangle \otimes |n\rangle - |p_2\rangle \otimes |-n\rangle),$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}} (|p_2\rangle \otimes |n\rangle + |p_1\rangle \otimes |-n\rangle),$$

$$|\psi_4\rangle = \frac{1}{\sqrt{2}} (|p_2\rangle \otimes |n\rangle - |p_1\rangle \otimes |-n\rangle).$$

(2.3)

Here, $|\pm n\rangle$ are the Bloch sphere representation of the spin state (qubit) as

$$|n\rangle = \begin{pmatrix} \cos\frac{\xi}{2} \\ e^{i\tau} \sin\frac{\xi}{2} \end{pmatrix}, \quad |\pm n\rangle = \begin{pmatrix} \sin\frac{\xi}{2} \\ \mp e^{i\tau} \cos\frac{\xi}{2} \end{pmatrix},$$

(2.4)

where $\xi$ and $\tau$ are polar and azimuthal angles, respectively.

2.1. Relativistic single spin $1/2$ particle quantum states

We assumed that spin and momentum are in the $yz$ plane [$\tau = \pi/2$;
$p = (0, p \sin \theta, p \cos \theta)$] and the Lorentz boost is orthogonal to it. For an observer in
another reference frame $S'$ described by an arbitrary boost $\Lambda$ in the $x$ direction, the
transformed $BD$ states are given by (see App. A)

$$|\psi_i\rangle \rightarrow U(\Lambda)|\psi_i\rangle,$$

$$|\Lambda \psi_1\rangle = \frac{1}{\sqrt{2}} \left( |\Lambda p_1\rangle \otimes \begin{pmatrix} \cos \frac{\xi}{2} \cos \frac{\Omega_{p_1}}{2} - i \sin \frac{\Omega_{p_1}}{2} \sin \frac{\zeta}{2} \\ i \sin \frac{\xi}{2} \cos \frac{\Omega_{p_1}}{2} + \sin \frac{\Omega_{p_1}}{2} \cos \frac{\zeta}{2} \end{pmatrix} + |\Lambda p_2\rangle \otimes \begin{pmatrix} \sin \frac{\xi}{2} \cos \frac{\Omega_{p_2}}{2} - i \sin \frac{\Omega_{p_2}}{2} \cos \frac{\zeta}{2} \\ -i \cos \frac{\xi}{2} \cos \frac{\Omega_{p_2}}{2} - \sin \frac{\Omega_{p_2}}{2} \sin \frac{\zeta}{2} \end{pmatrix} \right),$$

$$|\Lambda \psi_2\rangle = \frac{1}{\sqrt{2}} \left( |\Lambda p_1\rangle \otimes \begin{pmatrix} \cos \frac{\xi}{2} \cos \frac{\Omega_{p_1}}{2} - i \sin \frac{\Omega_{p_1}}{2} \sin \frac{\zeta}{2} \\ i \sin \frac{\xi}{2} \cos \frac{\Omega_{p_1}}{2} + \sin \frac{\Omega_{p_1}}{2} \cos \frac{\zeta}{2} \end{pmatrix} - |\Lambda p_2\rangle \otimes \begin{pmatrix} \sin \frac{\xi}{2} \cos \frac{\Omega_{p_2}}{2} - i \sin \frac{\Omega_{p_2}}{2} \cos \frac{\zeta}{2} \\ -i \cos \frac{\xi}{2} \cos \frac{\Omega_{p_2}}{2} - \sin \frac{\Omega_{p_2}}{2} \sin \frac{\zeta}{2} \end{pmatrix} \right),$$

$$|\Lambda \psi_3\rangle = \frac{1}{\sqrt{2}} \left( |\Lambda p_2\rangle \otimes \begin{pmatrix} \cos \frac{\xi}{2} \cos \frac{\Omega_{p_2}}{2} + i \sin \frac{\Omega_{p_2}}{2} \sin \frac{\zeta}{2} \\ i \sin \frac{\xi}{2} \cos \frac{\Omega_{p_2}}{2} - \sin \frac{\Omega_{p_2}}{2} \cos \frac{\zeta}{2} \end{pmatrix} + |\Lambda p_1\rangle \otimes \begin{pmatrix} \sin \frac{\xi}{2} \cos \frac{\Omega_{p_1}}{2} + i \sin \frac{\Omega_{p_1}}{2} \sin \frac{\zeta}{2} \\ -i \cos \frac{\xi}{2} \cos \frac{\Omega_{p_1}}{2} + \sin \frac{\Omega_{p_1}}{2} \sin \frac{\zeta}{2} \end{pmatrix} \right),$$

$$|\Lambda \psi_4\rangle = \frac{1}{\sqrt{2}} \left( |\Lambda p_2\rangle \otimes \begin{pmatrix} \cos \frac{\xi}{2} \cos \frac{\Omega_{p_2}}{2} + i \sin \frac{\Omega_{p_2}}{2} \sin \frac{\zeta}{2} \\ i \sin \frac{\xi}{2} \cos \frac{\Omega_{p_2}}{2} - \sin \frac{\Omega_{p_2}}{2} \cos \frac{\zeta}{2} \end{pmatrix} - |\Lambda p_1\rangle \otimes \begin{pmatrix} \sin \frac{\xi}{2} \cos \frac{\Omega_{p_1}}{2} + i \sin \frac{\Omega_{p_1}}{2} \sin \frac{\zeta}{2} \\ -i \cos \frac{\xi}{2} \cos \frac{\Omega_{p_1}}{2} + \sin \frac{\Omega_{p_1}}{2} \sin \frac{\zeta}{2} \end{pmatrix} \right),$$

where $\zeta = (\xi - 2\theta)$ and \{|\Lambda p_1\rangle, |\Lambda p_2\rangle\} are two orthogonal momentum eigenstates after Lorentz transformation.

The $BD$ density matrix (2.2), which describes the state of the single particle in the nonrelativistic frame, is exchanged with the density matrix $\rho'$ after Lorentz
transformation, i.e.

\[ \rho \rightarrow U(\Lambda)\rho, \]

\[ \rho' = U(\Lambda)\rho = \sum_{i=1}^{4} P_i |\Lambda \psi_i\rangle \langle \Lambda \psi_i|, \]  

(2.6)

It can be calculated that \(|\psi_i\rangle\) will be orthogonal after Lorentz transformation, i.e.

\[ \langle \Lambda \psi_i | \Lambda \psi_j \rangle = \delta_{ij}. \]

3. Spin–Momentum Correlation

We know that a system is entangled when its density matrix cannot be written as a convex sum of product states. For a pure state, dividing the system into two subsystems, A and B, allows the von Neumann entropy to be used as a measure of entanglement that corresponds to Ref. 6 being not Lorentz-invariant. When a bipartite system is in a mixed state, there are a number of proposals for measures of the entanglement of it, including entanglement of formation,\(^{19-22}\) relative entropy of entanglement\(^{23}\) and distillation of entanglement.\(^{24}\) For pure states each of these reduces to the von Neumann entropy. The best-known bipartite measure of entanglement is entanglement of formation. Because of this, we apply the concurrence (introduced by Wootters) related to entanglement of formation to measure the mixed-state entanglement of spin–momentum in the inertial frame.

3.1. Spin–Momentum Correlation of a Pure State

We show that by the von Neumann entropy, the entanglement for a pure state in the Schmidt form\(^{25}\) is not invariant after Lorentz transformation, and depends on the angles between spin and momentum. We introduce the pure state

\[ |\psi\rangle = \sqrt{\lambda_1} |n\rangle \otimes |p_1\rangle + \sqrt{\lambda_2} |n\rangle \otimes |p_2\rangle, \]  

(3.7)

where \(\lambda_1 + \lambda_2 = 1\).

We take the trace over the momentum eigenstates and obtain the reduced spin density matrix

\[ \rho' = \bigotimes_{\Delta p_1, \Delta p_2} (|\Lambda \psi\rangle \langle \Lambda \psi|), \]

with the two different eigenvalues

\[ \eta_1 = \frac{1}{2} \left\{ \lambda_1 + \lambda_2 - \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_1 \lambda_2 (\cos 2\varphi - 2\cos^2 \varphi \cos(\Omega_{p_1} - \Omega_{p_2}) - 1)} \right\}, \]

\[ \eta_2 = \frac{1}{2} \left\{ \lambda_1 + \lambda_2 + \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_1 \lambda_2 (\cos 2\varphi - 2\cos^2 \varphi \cos(\Omega_{p_1} - \Omega_{p_2}) - 1)} \right\}, \]  

(3.8)
where $\varphi$ is the angle between spin and momentum ($\varphi = \xi - \theta$). After some mathematical manipulations we have (see App. B)

$$E(\rho') \leq E(\rho). \quad (3.9)$$

This shows that the inequality (3.9) indicates that when the Lorentz boost and momentum are perpendicular, spin–momentum entanglement decreases with increasing velocity of the observer, as well as when spin and momentum are perpendicular, i.e.

$$\varphi = \frac{\pi}{2} \Rightarrow \eta_1 = \lambda_1, \quad \eta_2 = \lambda_2,$$

which shows that Lorentz transformation does not change the entanglement between them, i.e. $E(\rho') = E(\rho)$.

### 3.2. Spin–momentum entanglement of a mixed state

This subsection is devoted to calculating the concurrence of a relativistic $BD$ mixed state which is given in (2.6). By using App. A, we obtain the following result:

$$\lambda_1 = \frac{1}{2\sqrt{2}} \left\{ \sqrt{A_1 + B_1 - \sqrt{C_1 D_1}} \right\},$$

$$\lambda_2 = \frac{1}{2\sqrt{2}} \left\{ \sqrt{A_1 + B_1 + \sqrt{C_1 D_1}} \right\},$$

$$\lambda_3 = \frac{1}{2\sqrt{2}} \left\{ \sqrt{A_2 + B_2 - \sqrt{C_2 D_2}} \right\},$$

$$\lambda_4 = \frac{1}{2\sqrt{2}} \left\{ \sqrt{A_2 + B_2 + \sqrt{C_2 D_2}} \right\},$$

where

$$A_{(1)} = 3P_{(2)}^2 + 3P_{(4)}^2 - (P_{(2)}^2 + P_{(4)}^2) \cos 2\varphi,$$

$$B_{(1)} = 2\cos^2 \varphi (2P_{(1)} \Omega_{(4)} + (P_{(1)} - P_{(4)})^2 \cos \omega),$$

$$C_{(1)} = (P_{(1)} - P_{(4)})^2 (3 + \cos 2\varphi - 2\cos^2 \varphi \cos \omega),$$

$$D_{(1)} = (-3P_{(2)} + P_{(4)}) (P_{(2)} + 3P_{(4)})$$

$$+ (P_{(1)} - P_{(4)})^2 (\cos 2\varphi - 2\cos^2 \varphi \cos \omega),$$

where $\lambda_i$ are the square roots of the eigenvalues $\rho'$ and $\omega = \Omega_{(1)} + \Omega_{(4)}$. The first index (1) in ($A_{(1)}$, $B_{(1)}$, $C_{(1)}$, $D_{(1)}$) corresponds to ($P_{(1)}$, $P_{(2)}$) and the second index (2) corresponds to ($P_{(1)}$, $P_{(4)}$). Therefore

$$C(\rho') = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} \quad (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4). \quad (3.10)$$

To see the behavior of concurrence with respect to the boost in the $z$ direction, after some calculation we obtain the following results:

$$(\lambda_1 - \lambda_2)^2 = (P_{(2)} - P_{(4)})^2 \left(1 - \cos^2 \varphi \sin^2 \frac{\omega}{2}\right),$$

$$(\lambda_1 + \lambda_2)^2 = (P_{(2)} + P_{(4)})^2 - (P_{(2)} - P_{(4)})^2 \cos^2 \varphi \sin^2 \frac{\omega}{2}. \quad (3.11)$$
Using Eqs. (3.11), we obtain

\[
(\lambda_3 - \lambda_4) - (\lambda_1 + \lambda_2) = (P_1 - P_4) \sqrt{1 - \cos^2 \varphi \sin^2 \frac{\omega}{2}}
\]

\[
- \sqrt{(P_2 + P_3)^2 - (P_2 - P_3)^2 \cos^2 \varphi \sin^2 \frac{\omega}{2}}
\]

It is easy to see that

\[
(P_2 + P_3)^2 - (P_2 - P_3)^2 \cos^2 \varphi \sin^2 \frac{\omega}{2} \geq (P_2 + P_3)^2 \left(1 - \cos^2 \varphi \sin^2 \frac{\omega}{2}\right)
\]

so we have

\[
(\lambda_3 - \lambda_4) - (\lambda_1 + \lambda_2) \leq (P_1 - P_4) \sqrt{1 - \cos^2 \varphi \sin^2 \frac{\omega}{2}}
\]

\[
- (P_2 + P_3) \sqrt{1 - \cos^2 \varphi \sin^2 \frac{\omega}{2}}
\]

\[
= (P_1 - P_4 - P_2 - P_3) \sqrt{1 - \cos^2 \varphi \sin^2 \frac{\omega}{2}}
\]

\[
\leq (P_1 - P_4 - P_2 - P_3)
\]

and therefore

\[
C(\rho') \leq C(\rho).
\]

This shows that the spin–momentum correlation in the single-particle mixed quantum state is dependent on the angle between spin and momentum. Likewise, when spin and momentum are perpendicular, i.e. \(\varphi = \pi/2\), the concurrence is not an observer-dependent quantity in the inertial frame, namely \(C(\rho') = C(\rho)\).

4. Manipulating Quantum Control Gates via Lorentz Transformation

We explain how the Lorentz transformations can be realized as quantum control gates. To do this, we consider the pure state of (3.7) under Lorentz transformations as

\[
U(\Lambda)|\psi\rangle = \sqrt{\lambda_1}|\Lambda p_1\rangle \otimes W(n_1, p_1)|n_1\rangle + \sqrt{\lambda_2}|\Lambda p_2\rangle \otimes W(n_2, p_2)|n_2\rangle,
\]

where \(W(n_k, p_j)\) is Wigner rotation and the spinors are rotated by the Wigner angles. As a result, the Wigner rotation essentially behaves like a quantum control gate or controlling operator with the control quantum states \(\{|p_1\rangle, |p_2\rangle\}\) and target states \(\{|n_1\rangle, |n_2\rangle\}\). In order to better see the quantum control gate, we assume that the reference frame \(S'\) is described by an arbitrary Lorentz boost in the \(x\) direction and momentum and spin are parallel in the \(z\) direction, i.e. \(\varphi = 0\). Then the transformed
states in $2 \otimes 2$ Hilbert space are given by

\begin{align*}
|p_1\rangle \otimes \frac{1}{2} & \rightarrow \cos \frac{\Omega_{p_2}}{2} |\Lambda p_1\rangle \otimes \frac{1}{2} + \sin \frac{\Omega_{p_2}}{2} |\Lambda p_1\rangle \otimes -\frac{1}{2}, \\
|p_2\rangle \otimes -\frac{1}{2} & \rightarrow -\sin \frac{\Omega_{p_2}}{2} |\Lambda p_1\rangle \otimes \frac{1}{2} + \cos \frac{\Omega_{p_2}}{2} |\Lambda p_1\rangle \otimes -\frac{1}{2}, \\
|p_2\rangle \otimes \frac{1}{2} & \rightarrow \cos \frac{\Omega_{p_2}}{2} |\Lambda p_2\rangle \otimes \frac{1}{2} + \sin \frac{\Omega_{p_2}}{2} |\Lambda p_2\rangle \otimes -\frac{1}{2}, \\
|p_2\rangle \otimes -\frac{1}{2} & \rightarrow -\sin \frac{\Omega_{p_2}}{2} |\Lambda p_2\rangle \otimes \frac{1}{2} + \cos \frac{\Omega_{p_2}}{2} |\Lambda p_2\rangle \otimes -\frac{1}{2},
\end{align*}

where $\Lambda$, as the matrix representation of the Lorentz transformation in the computational basis \{|\Lambda p_1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, |\Lambda p_2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\}, is calculated as

\[
\Lambda = \begin{pmatrix}
\cos \frac{\Omega_{p_2}}{2} & \sin \frac{\Omega_{p_2}}{2} & 0 & 0 \\
-\sin \frac{\Omega_{p_2}}{2} & \cos \frac{\Omega_{p_2}}{2} & 0 & 0 \\
0 & 0 & \cos \frac{\Omega_{p_2}}{2} & \sin \frac{\Omega_{p_2}}{2} \\
0 & 0 & -\sin \frac{\Omega_{p_2}}{2} & \cos \frac{\Omega_{p_2}}{2}
\end{pmatrix}
\]

In the special case where $\Omega_{p_2} + \Omega_{p_1} = \pi$, we obtain

\[
\cos \frac{\Omega_{p_2}}{2} = \sin \frac{\Omega_{p_2}}{2}, \quad \sin \frac{\Omega_{p_2}}{2} = \cos \frac{\Omega_{p_2}}{2},
\]

and in the limit of $\Omega_{p_1} \rightarrow 0$ we get

\[
\lim_{\Omega_{p_1} \rightarrow 0} \Lambda = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0
\end{pmatrix}.
\] 

(4.13)

We know that the controlled-not (CNOT) gate is a two-qubit circuit that transforms the target qubit from its initial eigenstate into the opposite basis state iff the "control" qubit is in the eigenstate $|\frac{1}{2}\rangle$. Obviously, the quantum operation (4.13) flips the spin states, when the control momentum state is $|p_2\rangle$, and so the matrix representation (4.13) is similar to the CNOT gate. This CNOT is a nonlocal operation, because it can actually create a maximally entangled state from a product state or vice versa. For instance, after applying the gate (4.13) on the product state

\[
\left|\begin{array}{c}
\frac{1}{\sqrt{2}}
\end{array}\right.\left|\begin{array}{c}
\frac{1}{\sqrt{2}}
\end{array}\right.\]
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\((|p_1\rangle + |p_2\rangle) \otimes |1/2\rangle\), we obtain the entangled state

\[ (|p_1\rangle + |p_2\rangle) \otimes \frac{1}{\sqrt{2}} \rightarrow |\Lambda p_1\rangle \otimes \frac{1}{\sqrt{2}} + |\Lambda p_2\rangle \otimes -\frac{1}{\sqrt{2}} \]  
(4.14)

and for maximally entangled Bell state

\[ \frac{1}{\sqrt{2}} \left( |p_1\rangle \otimes \frac{1}{\sqrt{2}} + |p_2\rangle \otimes -\frac{1}{\sqrt{2}} \right) \rightarrow \frac{1}{\sqrt{2}} (|\Lambda p_1\rangle - |\Lambda p_2\rangle) \otimes \frac{1}{\sqrt{2}}, \]  
(4.15)

which is a separable state.

5. Conclusion

In this paper, we have considered the spin–momentum correlation in massive single spin-1/2 particle quantum states which furnish an irreducible representation of the Poincaré group. Instead of the superposition of all momenta, we considered only two momentum eigenstates \((p_1\) and \(p_2)\). We showed that the spin–momentum correlation in the relativistic single spin-1/2 particle mixed state (when the momentum is perpendicular to the boost direction) is dependent on the angle between spin and momentum and when they are parallel the measure of entanglement decreases with increasing velocity of the observer. We also showed that the Lorentz transformations can be realised as quantum control gates and they become like the CNOT gate in the limit where \(\beta \rightarrow 1\).

Appendix A

A.1. Wigner representation for spin-1/2

In Ref. 26, it is shown that the effect of an arbitrary Lorentz transformation \(A\) unitarily implemented as \(U(\Lambda)\) on single-particle states is

\[ U(\Lambda)(|\sigma\rangle) = \sqrt{\frac{(\Lambda p)^a}{p^b}} \sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda, p))(|\sigma\rangle \otimes |\sigma'\rangle), \]  
(A.1)

where

\[ W(\Lambda, p) = L^{-1}(\Lambda p)\Lambda L(p) \]  
(A.2)

is the Wigner rotation.\(^7\) We will consider two reference frames in this work: one is the rest frame \(S\) and the other is the moving frame \(S'\), in which a particle whose four-momentum \(p\) in \(S\) is seen as boosted with the velocity \(v\). By setting the boost and particle moving directions in the rest frame to be \(\hat{v}\) with \(\hat{v}\) as the normal vector in the boost direction and \(\hat{p}\), respectively, and \(\hat{n} = \hat{e} \times \hat{p}\), the Wigner representation for the spin-1/2 particle is found as\(^12\)

\[ D^{\dagger}(W(\Lambda, p)) = \cos \frac{\Omega p}{2} + i \sin \frac{\Omega p}{2} \left( D_\perp \right), \]  
(A.3)

\[ \text{Chir. dot} \]
where
\[
\cos \frac{\Omega_p}{2} = \frac{\cosh \frac{\alpha}{2} \cosh \frac{\delta}{2} + \sinh \frac{\alpha}{2} \sinh \frac{\delta}{2} (\hat{e} \cdot \hat{p})}{\sqrt{\frac{1}{2} + \frac{1}{2} \cosh \alpha \cosh \delta + \frac{1}{2} \sinh \alpha \sinh \delta (\hat{e} \cdot \hat{p})}},
\] (A.4)

\[
\sin \frac{\Omega_p}{2} \hat{n} = \frac{\sinh \frac{\alpha}{2} \sinh \frac{\delta}{2} (\hat{e} \times \hat{p})}{\sqrt{\frac{1}{2} + \frac{1}{2} \cosh \alpha \cosh \delta + \frac{1}{2} \sinh \alpha \sinh \delta (\hat{e} \cdot \hat{p})}},
\] (A.5)

and
\[
\cosh \alpha = \gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad \cosh \delta = \frac{E}{m}, \quad \beta = \frac{v}{c}.
\]

Appendix B

B.1. Entanglement of formation

Let \(|\psi\rangle = \sum_{i,j=1}^{N} a_{ij} e_i \otimes e_j, a_{ij} \in C\), be a two-particle pure state with normalization \(\sum_{i,j=1}^{N} |a_{ij}|^2 = 1\). For this pure state the entanglement of formation \(E\) is defined as the entropy of either of the two sub-Hilbert spaces, i.e.
\[
E(|\psi\rangle) = -\Tr(\rho_1 \log_2 \rho_1) = -\Tr(\rho_2 \log_2 \rho_2),
\] (B.1)

where \(\rho_1\) (respectively, \(\rho_2\)) is the partial trace of \(|\psi\rangle \langle \psi|\) over the first (respectively, second) Hilbert space. A given density matrix \(\rho\) on \(\mathcal{H}^d \otimes \mathcal{H}^d\) has pure-state decompositions of \(|\psi_i\rangle\) of the form (2.2) with probabilities \(P_i\). The entanglement of formation for the mixed state \(\rho\) is defined as the average entanglement of the pure states of the decomposition, minimized over all possible decompositions of \(\rho\), i.e.
\[
E(\rho) = \min \sum_i P_i E(|\psi_i\rangle).
\] (B.2)

In the case of \(n=2\), (B.1) can be written as
\[
E(|\psi\rangle)|_{n=2} = H \left( \frac{1 + \sqrt{1 - C^2}}{2} \right),
\] (B.3)

where \(H(x) = -x \log_2 x -(1-x) \log_2 (1-x)\) is binary entropy and \(C\) is called concurrence. Thus, calculation of (B.2) can be reduced to calculating the corresponding minimum of
\[
C(\rho) = \min_k \sum_{b=1}^{k} p_b C(|\psi_b\rangle).
\]

Wootters\(^{30}\) has shown that for a two-qubit system entanglement of formation of a mixed state \(\rho\) can be defined as
\[
E(\rho) = H \left( \frac{1 + \sqrt{1 - C^2}}{2} \right),
\] (B.4)
by

$$C(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4),$$

where \(\lambda_i\) are the nonnegative eigenvalues, in decreasing order, of the Hermitian matrix

$$R = \sqrt{\sqrt{\rho} \rho \sqrt{\rho}},$$

with

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y),$$

where \(\rho^*\) is the complex conjugate of \(\rho\) when it is expressed in a fixed basis such as \(|1\rangle, |\rangle\rangle\), and \(\sigma_y\) is

$$\begin{pmatrix}
0 & -i \\
 i & 0
\end{pmatrix}$$
on the same bases.

In order to obtain the concurrence of BD states \((2, 2)\), we use the method presented by Wootters. We define subnormalized orthogonal eigenvectors \(|\psi_i\rangle\) as

$$|\psi_i\rangle = \sqrt{\tau_i} |\psi_i\rangle, \quad \langle \psi_i | \psi_i \rangle = \tau_i \delta_{ij},$$

and define \(|x_i\rangle\) as

$$|x_i\rangle = \sum_{j=1}^{4} \tau_{ij}^* |\psi_i\rangle$$

for \(i = 1, 2, 3, 4\) such that

$$\langle x_i | x_j \rangle = \langle U^T \rho U \rangle_{ij} = \lambda_i \delta_{ij},$$

where \(\tau_{ij} = \langle \psi_i | \psi_j \rangle\) is a symmetric but not necessarily Hermitian matrix. In the construction of \(|x_i\rangle\) we have considered the fact that for any symmetric matrix \(\tau\) one can always find a unitary matrix \(U\) in such a way that \(\lambda_i\) are real and nonnegative, i.e. they are the square roots of eigenvalues of \(\tau^* \tau\) which are the same as the eigenvalues of \(R\). Moreover, one can always find \(U\) such that \(\lambda_1\) is the largest one. After some calculations we get the values for \(\lambda_i\)

$$\lambda_1 = P_1, \quad \lambda_2 = P_2, \quad \lambda_3 = P_3, \quad \lambda_4 = P_4,$$

and concurrence can be evaluated as

$$C(\rho) = (P_1 - P_2 - P_3 - P_4).$$

References