Optimal portfolio selection in a Value-at-Risk framework: 
Markov-Switching GARCH Approach

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Abstract
The aim of this paper is to calculate VaR index with parametric Markov-Switching GARCH approach for accepted companies in Tehran Stock Exchange and also selecting the optimal portfolio of their stocks. To calculate the index, data and information of weekly stock price of 10 representative firms over the period 2008-2014 has been used which account for 332 working weeks.
The results from estimation of VaR and determination of optimal stock portfolio in the non-linear programming framework showed that optimal portfolio of food-industry companies stock, in the context of VaR has higher returns in the first regime (Boom period) compared to the second regime (recession period), but had a lower risk. On the other hand, it has had lower weight in both stock portfolios that had lower average returns compared to the rest of the stocks and compared to the stocks which had lower VaR relative to other stocks that has higher weights.

Keywords: Optimal stock portfolio, Value at Risk (VaR), Food-industry companies, MS-GARCH approach.
1. Introduction

Investment has been one of the most important topics in all countries' economies both for individuals and civil persons in micro levels and for the economy authorities in macro levels. So that in the last two decades, development of financial markets and providing financial instruments have been one of the suitable way of absorbing more capital in international levels. But, in spite of new financial instruments (e.g. derivatives), and their wide applications in developed countries, still most of the developing countries especially Islamic ones has not been beneficiary of such instruments for several reasons. Therefore, traditional financial markets such as stock market form the main structure of these countries financial markets, such as Iran and have accounted for most of the investors and speculators in both first and secondary markets.

On the other side, emerging of liberalization back grounds and deregulation in some countries in recent decades whose initial signal was the abolishment of fixed-exchange rate regime in the early 1970's resulted in more fluctuations in price variables to be faced by financial markets. Also emerging of monetary unions and globalization of economies have resulted in an increase in spread of financial crisis from one market to another which means an increase in financial market fluctuations. On the other side, intense decrease of stock prices in most of the financial markets has resulted in considerable loss for the market activists.

Therefore activity in financial markets is accompanied by risk and uncertainty and measuring of risk in different portfolio is important for investors. Through this, exploring and measuring undesirable risks for financial markets and capital market activists has a significant importance.

Value at Risk is an index for undesirable risks and a measure for calculating the maximum probable loss in the portfolio which was provided by Weather Stone in 1994. VaR measures risk quantitatively and is now one of the common key instruments in risk management. By definition, VaR is the maximum loss which is a decrease in the value of the portfolio for a certain period in future and will not exceed that loss with a specific confidence level. In other words, VaR measure is the worst expected loss in the normal market conditions in an specific time period with a confidence level.

In line what is discussed above, the main goal of this research is to calculate the value at risk index with Markov-Switching approach for accepted food-industry companies stocks in Tehran Stock Exchange and selection of the optimal stock portfolio for those companies. In calculating the index, data and information of ten representative companies for the period 2008-2014 is used. In the preceding parts following the introduction, in the second part, theoretical fundamentals are noted. In the third part, a review of empirical studies is shown. The fourth part of the research introduces data and research methodology. The fifth part of the research includes analysis of the empirical results and in the last part the ultimate results are provided.

2. Literature Review

The researches that have been done on financial markets have shown that return distribution is not normal in these markets. The theory of determination of optimal portfolio based on undesirable risk was discussed. This theory considers a clear difference between desirable and undesirable fluctuations. In this theory just the fluctuations which are lower that objective return of the investors account for risk, whilst all fluctuations higher than that (in the case if uncertainty) are included as opportunities to gain desirable return. In a better way, this theory based on the relationship between returns and undesirable risk, defines investor's behavior a measure f optimal portfolio determination (Estrada, 2007).
In this theory, measures of undesirable risk are used as return risk index. Semi variance and semi beta indices are the most popular measures. But the theory of determination of optimal portfolio in the VaR framework is the most important sub group of this theory. Measure of VaR which is from a group of undesirable risk measurements has had a wide application in optimal portfolio studies during recent years.

2.1. History and concept of Value at Risk
The term "Value at Risk" has not been in finance literature until early 1990's, but the starting point of considering Value at Risk originates in several years ago, 1993 in which New York Stock Exchange for the first time unofficially examined the member companies' capital.

Albeit, the concept of Value at Rist firstly was suggested by Bamoul in 1963, that was three decades before its wide application, while exploring a model of "Expected gain confidence Unit Criterion"(Bamoul 1963). However, it could be said that from a more general point of view that "Safety Models" of professors of finance such as Roy in 1952 and Telse in 1955 have been the introduction of formation of VaR models.

Guldimann could be considered as the inventor of the term Value at Risk (Guldimann, 2000). In the late 1980's, he was the research manager of G.P Moragn bank. The risk management groups should decide on whether to choose the risk free investment in bonds and permanent income producing or investing in stocks and foreign exchange to keep the market value constant. The bank concluded that Value at Risk is more important than Earning Risk. This caused that the bank prepared a research group to explore through risk.

At that time, a lot of attention was paid to the risk of derivatives. Group 30 (G-30), in which G.P Morgan representative also were, began the topic of the best risk management methods. The term "Value at Risk" in July 1993, found its way in G-30 report. This was the first time that the term Value at Risk was widely applied. The other names of Value at Risk have been Capital at Risk and Dollars at Risk that were applied for some time. Considering the history, the consensus view in finance literature is that Value at Risk is a new approach for control and management of risk.

During recent years, the measure of Value at Risk has been highly accepted and popular among investors and finance sector activists. The reason for popularity and generality of this method is its simplicity in creating a statistical form of a summary of portfolio losses in a certain time horizon (Mohamed, 2005).

Although a change in value of a portfolio can be related to various risk elements, Value at Risk tries to estimate the decrease of portfolio value from market risk point of view. Market risk includes uncertainty of future incomes because of changes in market condition (prices and rates) (Kormasm, 1998).

In fact, Value at Risk is designed to provide a specific figure to the analyst and in that figure there exists information about portfolio risk. This measure is an underlying estimate of loss level in a portfolio of investment that is forecasted with a small specific probability that equals it or doesn't exceed. Value at Risk in contrast to traditional measures of risk, provides a general and comprehensive portfolio risk. As a result, Value at Risk, in fact is measuring of risk with futurist view that could be applied for all financial documents. The value at risk model includes three major elements of time horizon, level of confidence and the capital level (Dowd et.al, 2003)

2.2. Portfolio optimization problem
Two major elements in decision making for investment are risk and return of capital equities. Markowitz with his mean-variance model showed that by constructing a portfolio of financial equities, it would be possible to decrease risk in a certain level of return. This possibility exists because of lack of complete
correlation among financial equity returns. Different individuals invest based on their expected utility and decrease their today's consumption with an expectation of more consumption in future. Utility function of each investor is determined based on his own preferences and is not necessarily identical with the others (Raie and Alibeygi, 2010).

Risk and return are measures that determine level of utility of the investor from his equities. Selecting optimal equity is mostly done by trade-off between risk and return and the more is the risk of sort of equity, investors expect to receive higher returns. Identifying the efficient frontier of portfolio provides the possibility for the investors to gain maximum expected return of investment based on his utility function and degree of risk loving and risk aversion. Each of the investors according to their degree of risk aversion chooses a point on the efficient frontier and determines their portfolio blend with the goal of maximization of returns and minimization of risk (Raie and Talangi, 2004).

Portfolio optimization is selection of the best blend of financial equities that results in maximization of return and minimization of risk with the highest probability. The fundamental idea of Modern Portfolio Theory, MPF, is that if we invest in the equities that don't have complete correlation, risk of that equities eliminate themselves, therefore a constant return with a lower risk can be achieved (Markowitz, 1952).

In general, in Financial Economics literature, two theories are more considered: Modern Portfolio Theory and Theory of optimal portfolio determination based on undesirable risk measures. In Modern Portfolio Theory, optimal allocation of equities and identifying optimal portfolio is done based on mean and variance of return. (Mean-Variance Optimization, MOV). In the other theory, optimal allocation of equities and identifying optimal portfolio is done based on downside relation of returns and undesirable risk measures (Downside Risk Optimization, DRO).

Considering that in the current research optimization of stock portfolio is done under the VaR constraint so will be explained in the proceeding parts.

In the VaR approach for optimal portfolio selection, the basics are similar to Markowitz with this difference that the investor seeks lower VaR and higher return (Campbell et al., 2001). In order to reach to the optimal portfolio (optimal weights of each stock and optimal VaR) the problem (1) should be solved:

\[
\begin{align*}
\text{Min} & \quad \text{VaR}_p \\
\text{S.t.} & \\
\sum_{i=1}^{n} w_i &= 1 \\
\sum w_i \bar{R}_i &\geq R^* \\
w_i &\geq 0 \quad i = 1, ..., n
\end{align*}
\]

In which, \( \text{VaR}_p \) is the VaR of portfolio and all necessary information such as VaR of each stock, average return of each stock and return of the whole portfolio are available and the \( w_i \)s are unknown. Optimal VaR for the whole portfolio of stocks can be calculated by the below equation:
After solving the above problem, with non-linear programming (NLP), the efficient frontier of investment will be achieved. Generally, efficient frontier is in the form in below:

\[
\text{VaR}_p = MZ_a \sigma_p = MZ_a \sqrt{\sum_{i=1}^{9} \sum_{j=1}^{9} w_i w_j \text{cov}(i, j)} = MZ_a \sqrt{\sum_{i=1}^{9} w_i^2 \sigma_i^2 + \sum_{i=1}^{9} \sum_{j<i}^9 w_i w_j \rho_{ij} \sigma_i \sigma_j}
\]

\[
= \sum_{i=1}^{9} (w_i \sigma_i MZ_a)^2 + \sum_{i=1}^{9} \sum_{j<i}^9 w_i w_j \rho_{ij} \sigma_i \sigma_j (MZ_a)^2 = \sqrt{\sum_{i=1}^{9} w_i^2 \text{VaR}_i^2 + \sum_{i=1}^{9} \sum_{j<i}^9 w_i w_j \text{VaR}_i \text{VaR}_j \rho_{ij}}
\]

After solving the above problem, with non-linear programming (NLP), the efficient frontier of investment will be achieved. Generally, efficient frontier is in the form in below:

The empirical literature on VaR has focused on value at risk calculate methodes or portfolio optimization. Different papers done such as Soni, 2005; Hung et al., 2007; Wu and Shieh, 2007; Dockery and Efentakis, 2008; Abad and Benito, 2009; Nieto and Ruiz, 2010; Yu et al., 2011.

3. Data and Methodology

3.1. Data

The population of this research is accepted food-industry companies (excluding from sugar) in Tehran Stock Exchange. According to the data gathered from Tehran Exchange website, up to the time of this study, this group included 20 companies. In order to research, the logarithmic return of these companies stock is used that is calculated via average weekly prices of stocks in the period Jan 2008 to May 2014, which accounts for 332 working weeks. After calculating the stock returns, one observation is eliminated and the period length decreases to 331 weeks. Data and information of weekly prices of companies are extracted from Rahavard-Novin Software of Iran Stock Exchange. It should be noted that in this study the
companies which their symbols have been closed for more than 6 months are eliminated from the study. So considering the given constraint, ten companies from the twenty are selected".

3.2. Methodology

Manganelli and Engle (2004), classify methods of VaR calculation into 3 groups, parametric, semi-parametric and non-parametric. Parametric approaches include parameterization of prices. In this approaches, a specific assumption regarding to the probability distribution of return with conditional quartiles using estimation of conditional fluctuations considering an assumption for distribution is estimated. GARCH models mostly are used for forecasting fluctuations (Poon and Granger, 2003).

The most important non-parametric calculation of VaR that is widely used in financial studies is Historical Simulation that does not need any specific assumption regarding to return distributions and estimates VaR as quartile and percentile of empirical distribution of historical returns from a mobile window of early periods (Taylor, 2008). One of the problems which exist in this approach is that a mobile window includes several historical periods. Having more or less of periods may result in sampling errors on bias in results. The semi-parametric approach also includes several methods that the most important one is Mont Carlo Simulation. The idea of Mont Carlo Simulation is the continuous random process upon price or return of the financial instruments.

Because in this paper the Makov-Switching GARCH approach is used to calculate the VaR, this method is explained in proceeding parts.

Let \( S_t \in \{1, \ldots, m\} \) denote the (unobserved) regime at a discrete time \( t \) and let \( s_t \) be a realization of \( S_t \)
assuming that is a first-order stationary Markov chain with transition probabilities \( a_{ij} = p(S_t = j | S_{t-1} = i) \), a transition probabilities matrix \( A \), \( A_{ij} = a_{ij} \) and stationary probabilities \( \pi = [\pi_1, \pi_2, \ldots, \pi_m] \) in which \( \pi_i = p(S_t = i) \) where \( \cdot^T \) denotes the transpose operation.

Incorporating GARCH models with a hidden Markov chain, where each state of the chain (regime) allows a different GARCH behavior and thus a different volatility structure, extends the dynamic formulation of the model and potentially enables improved forecasts of the volatility (Formmel, 2004). Unfortunately volatility of GARCH process with switching regimes depend on complete history of process including regime path, which makes the derivation of a volatility estimator impractical. Many kinds of Markov-Switching GARCH models are introduced up until now. In the study of Gray(1996), several kinds of MS-GARCH regarding to this assumption that conditional variance of a certain regime depends on the value of last year conditional variance not the current value has been introduced.

One the most famous MS-GARCH models in the Econometrics is a model by Klassen(2002) on the adjusted model of Gray(1996). These models disturb the path of unobservable regimes and eliminate the accumulation. Therefore, conditional variance can be achieved just based on previous observations. According to this, for the Markowitz state, \( s_t \in \{1, 2\} \) and the conditional variance would be (Klassen,2002):

\( \cdot^* \). List of companies repoted in appendix.
\[ \sigma_{t,s_t}^2 = w_{s_t} + \alpha_{s_t} e_{t-1}^2 + \beta_{s_t} E \{ \sigma_{t-1,s_{t-1}}^2 | s_{t-1}, \psi_{t-1} \} \]  

(3)

As is shown in equation (3), this model is Markov chain with two regimes with GARCH (1,1) in each. Matrix \( C_{2 \times 2} \) is defined by

\[
\begin{bmatrix} \sigma_1^2 \\ \sigma_2^2 \end{bmatrix} = (I - C)^{-1} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}
\]

(4)

What is used in the current research as a way of calculating value at risk can be explained this way:

Stock markets and stock fluctuations can be classified into two groups: \( s_t = 1 \) for the first regime and \( s_t = 2 \) for the second one. If \( r_t \) is the series of financial returns and follows a model of structural break, then we will have:

\[
r_t = \begin{cases} c_1 + \alpha_s x_t + u_t & \text{if } s_t = 1 \\ c_2 + \alpha_s x_t + u_t & \text{if } s_t = 2 \end{cases}
\]

(5)

In summary, \( r_t = C_{s_t} + \alpha_s x_t + u_t \), that \( u_t \sim N(0, \sigma_{s_t}^2) \) and \( x \) is an exogenous variable. In the present study just \( C_{s_t} \) is considered as conditional mean for the equation. Conditional variance equation also in the form of GARCH (1,1) is modeled with 2 regimes which is the suggested model of Klassen.

4. Empirical Results

The main goal of this study is to determine the optimal portfolio of food-industry countries accepted in Tehran Stock Exchange. After considering constraint of closing symbols, stocks of 10 companies in food-industry in order to determine the optimal portfolio is consolidated. Data and information of weekly prices of stocks of these companies form Jan 2008 to May 2014 which is 332 working weeks is extracted from Rahavard-Novin software and using return of these companies stocks are calculated as:

\[
r_i = (p_i - p_{i-1}) \times 100
\]

(6)

where \( p_i \) equals \( \ln(p_i) \) and \( P_i \) shows price at time \( t \). After calculating return of the stocks, one observation is eliminated and time period is decreased to 331 weeks. Table (1) represents descriptive characteristics of data on stock returns of the companies. In this table mean and standard deviation of each of the stocks is reported. It should be explained that because of closing of these companies, symbols during some weeks of the given period, number of observation for stock returns of studied companies are different:
According to the table (1), the highest mean return belongs to company number (2) and the lowest to the company number (1).

In the rest, the value of Value at Risk of stocks with the assumption of 95% confidence level and for a future period with MS-GARCH approach is forecasted. It should be noted that for all return series, the variance equation is eliminated in the GARCH (1,1) form with 2 regimes. Table (2) shows the VaR in the 95% confidence level for the two regimes.

**Table (2): Value at Risk Index for Return of the Food-Industry Companies**

<table>
<thead>
<tr>
<th>Company's stock</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>First Regime</strong></td>
<td>-0.087</td>
<td>-0.1305</td>
<td>-0.0879</td>
<td>-0.01305</td>
<td>-0.0948</td>
<td>-0.0608</td>
<td>-0.0772</td>
<td>-0.0802</td>
<td>-0.1006</td>
<td>-0.1069</td>
</tr>
<tr>
<td><strong>Second Regime</strong></td>
<td>0.0557</td>
<td>-0.0758</td>
<td>-0.0826</td>
<td>-0.0758</td>
<td>-0.0090</td>
<td>-0.0006</td>
<td>-0.0076</td>
<td>-0.0039</td>
<td>-0.0319</td>
<td>-0.0287</td>
</tr>
</tbody>
</table>

Estimation of model shows that high boom and high return periods are in first regime and recession and low return periods are in the second one. Considering the balance of risk and return, it is expected that the risk is lower in the second regime which is clearly seen in the above table.

It is worth to note that the VaR is presented in form of percentage in the table. For example, VaR of the first company shows that by investing in its stock, the maximum loss from that during the future week equals 8.7% investment.

Considering the average of stock returns as the expected return and VaR as the expected risk, the optimization problem in order to obtain efficient frontier of optimal portfolio will be:
\[
\text{Min } \text{VaR}_p = \sqrt{w_1^2 \text{VaR}_1^2 + w_2^2 \text{VaR}_2^2 + \ldots + w_{10}^2 \text{VaR}_{10}^2}
\]

s.t.:
\[
\sum_{i=1}^{10} w_i = 1
\]
\[
\sum_{i=1}^{10} w_i \bar{R}_i \geq R^*
\]
\[
w_i \geq 0 \quad i = 1, \ldots, 10
\]

After solving the above problem, with non-linear programming approach (NLP), the efficient frontier of investment or optimal portfolio in the VaR framework is like this. Figures (2) and (3) show efficient frontier or investment in the framework of return and VaR in first and second regimes respectively.

![Figure (2): Efficient Investment Frontier in the Expected Risk and return Framework (First regime)](image-url)
In the proceeding parts, investor preferences in the form of utility functions enter the analysis and optimal portfolio in the return-value at risk framework and considering the degree of risk aversion is determined. It should be explained that in the optimization programs. Interest rate is risk free, week interest on load is 0.4 and the degree of risk aversion of investor is considered 0.5. The utility function of the investor is assumed as $U = E(r) - \frac{1}{2} A \sigma^2$ where $E(r)$ is the expected return and $A$ and $\sigma^2$ respectively show the degree of risk aversion and VaR. figures (4) and (5), show optimal risky and compound portfolio and the optimal stock weight (stock weights in optimal portfolio) of food-industry companies are provided in table (3):

**Table (3): Optimal portfolio of Food-Industry Companies (Risky Portfolio)**

<table>
<thead>
<tr>
<th>stock weight</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>Portf olio return</th>
<th>Portf olio VaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Regime</td>
<td>0</td>
<td>0.2377</td>
<td>0.0458</td>
<td>0</td>
<td>0.0485</td>
<td>0.1039</td>
<td>0.2530</td>
<td>0.0573</td>
<td>0.0229</td>
<td>0.2310</td>
<td>0.0074</td>
<td>-0.0057</td>
</tr>
<tr>
<td>Second Regime</td>
<td>0</td>
<td>0.0253</td>
<td>0.0030</td>
<td>0</td>
<td>0.0315</td>
<td>0.6507</td>
<td>0.1590</td>
<td>0.0728</td>
<td>0.0044</td>
<td>0.0532</td>
<td>0.0057</td>
<td>-0.0246</td>
</tr>
</tbody>
</table>

Figure (3): Efficient Investment Frontier in the Expected Risk and return Framework (second Regime)
As the results show, in the optimal portfolio of an average risk averter investor, in the first regime, companies (1) and (4) (Khorak Dam Pars and Pars Minoo) are not placed in the optimal portfolio. Company (7), Pegah Isfahan has the highest share of the financial portfolio and because of this, companies (2) and (10) are in the next classification.

Figure (4): Optimal Risky and Compound Portfolio (including risky and risk-free assets) - Based on VaR (first regime)

Figure (5): Optimal Risky and Compound Portfolio (including risky and risk-free assets) - Based on VaR (second regime)
Also in the second regime, companies (1) and (4) are not placed in the optimal portfolio. The highest share is for the sixth company (Dasht-e-Morghab) which solely accounts for 65% of the portfolio. Stock of the company (7) is ranked in the next level. Stock of the companies (9) and (3) are near to zero and negligible.

Focusing on the weights obtained shows that the companies (1) and (4) as a result of having the lowest average return among other companies, are not placed in the optimal portfolio. On the other hand, the companies with lowest value at risk have the highest weight in the portfolio. For instance, in the second regime, company (6) has a negligible VaR and therefore has been entitled a considerable share of the optimal portfolio. Also the return of the boom portfolio is higher than the second one and its risk is lower in comparison to the risk of the portfolio of recession period.

5. Conclusion

The aim of this study is to determine the optimal portfolio of selected food-industry companies that are active in Tehran Stock Exchange in the framework of VaR calculated by parametric MS-GARCH approach. Therefore, value at risk is calculated with the data and information of weekly prices of stocks in the period 2008-2014 in 2 regimes with GARCH (1,1).

Then by using expected return and calculated VaR, the efficient frontier for investment and optimal ultimate portfolio of investors in the form of mean-Value at Risk model for both regimes is achieved. The results showed that optimal portfolios of food-industry companies in the value at risk framework calculated in the first regime have higher return compared to the second regime but the risk is lower. On the other hand, in both portfolios that had average returns lower weight and the stock that had lower VaR compared to the rest of the stocks, had higher weights.
References


Appendix

10 representative food-industry companies in Tehran stock Exchange

<table>
<thead>
<tr>
<th>Number</th>
<th>Company name</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Piadaz Agriculture and Industrial</td>
</tr>
<tr>
<td>2</td>
<td>Behshahr Industrial</td>
</tr>
<tr>
<td>3</td>
<td>Pak Diary</td>
</tr>
<tr>
<td>4</td>
<td>Pars Mioo</td>
</tr>
<tr>
<td>5</td>
<td>Khorak Dam Pars</td>
</tr>
<tr>
<td>6</td>
<td>Dasht-e-Morghab</td>
</tr>
<tr>
<td>7</td>
<td>Isfahan Pasteurized Milk- Pegah</td>
</tr>
<tr>
<td>8</td>
<td>Shahd Iran</td>
</tr>
<tr>
<td>9</td>
<td>Margarin</td>
</tr>
<tr>
<td>10</td>
<td>Behshahr Industries development</td>
</tr>
</tbody>
</table>

Reference: http://www.tse.ir