Exactly solvable model of transitional nuclei based on dual algebraic structure for the three level pairing model in the framework of sdg interacting boson model

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Abstract

In this paper, a successful algebraic method based on the dual algebraic structure for three level pairing model in the framework of sdg IBM is proposed for transitional nuclei which show transitional behavior from spherical to gamma-unstable quantum shape phase transition. In this method complicated sdg Hamiltonian, which is a three level pairing Hamiltonian is determined easily via the exactly solvable method. This description provides a better interpretation of some observables such as BE(4) in nuclei which exhibits the necessity of inclusion of g boson in the sd IBM, while BE(4) cannot be explained in the sd boson model. Some observables such as Energy levels, BE(2), BE(4), the two neutron separation energies signature splitting of the γ-vibrational band and expectation values of the g-boson number operator are calculated and examined for 104−110 Pd isotopes.
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1. Introduction

Description of the transitional nuclei as well as finding a simple method which can be useful in theoretical studies of these nuclei has become the most important subject of research in low-energy nuclear structure physics over last years. Indeed, the study of quantum phase transitions in finite nuclear systems has recently been the subject of many investigations. Several methods have been used, advocated, developed and extended. Among them are duality relations which provide a simple method for solving these problems. There has been growing evidence for the existence of different types of duality relations [1–9] in theoretical nuclear physics. The duality relations have proven to be a powerful tool in relating the Hamiltonians of number-nonconserving quasispin and the number-conserving unitary algebras for a system with pairing interactions. One of the algebraic formulations for description of finite pairing systems is quasispin algebra [10–12]. The nuclear pairing Hamiltonian [13–16], was solved by integrable models called Richardson–Gaudin (RG) models derived from adding the Richardson exact solution [17] to the integrable model proposed by Gaudin [18]. Richardson exact solution is an algebraic approach to the pairing problem and rather a complex formalism which was not widely used in nuclear physics. Indeed the quasispin formalism of Kerman [9] was applied to a limited class of seniority-conserving pairing interactions. Development of consideration of these interactions has been the focus of the recent works based on the algebraic Bethe ansatz and infinite-dimensional Lie algebraic methods [4–9]. In these models, the Hamiltonian is written as a linear combination of the Casimir operators of a group. By using such relations, calculations for two-level and multi-level systems can be easily explained [10,15,19].

Quantum Phase Transitions (QPTs) are abrupt changes in the structure of a physical system. Indeed, they describe the nuclear ground state and low-lying excitations associated with it. QPTs corresponds to two classes: a first-order QPT between spherical and axially deformed shapes, a second-order QPT between spherical and 2-soft shapes [20–25]. During the transition from one limit to another, meeting points in which their potential has a flat behavior. These points are called critical points introduced by Lachello in the framework of the collective model for even–even nuclei [24,25]. In nuclear physics, quantum phase transitions can be studied most easily using the algebraic techniques [26] such as interacting boson model [27–31] which provides an algebraic description of the quadrupole collective properties of low-lying states in nuclei in terms of a system of interacting bosons. The bosons are assumed to be made up of the correlated pairs of valence nucleons. In interacting boson model, nuclear collective excitations of even–even nuclei contain only $L = 0$ s-boson or $L = 2$ d-boson nucleon pairs and the nuclear structure is described within the U(6) symmetry, that includes rotational nuclei SU(3), vibrational U(5) and $\gamma$-unstable nuclei O(6) symmetry limits. These three limits are a good examples of exactly solvable problems. In these limits, analytic results for the energy spectra and the wave functions can be achieved with the help of the group representation theory. It is worth mentioning that, when the Hamiltonian is written in terms of invariant operators of the two or three limiting cases, diagonalizing the Hamiltonian by using numerical methods is necessary.

IBM has been rather successful in describing the collective properties, level scheme and electric quadrupole transition E(2) properties of several medium and heavy mass nuclei. However, when this model was used to compare the electric hexadecapole transitions E(4) in the nuclei with the experimental data, the results were not persuading shell-model calculation which found that $I = 0$, $2$ coupled fermion pairs constitute only about 70 percent of the wave functions. On the other hand, in many transitional nuclei near the closed shells such as Zr, Mo, Ru, Pd, Cd, Sn adjacent to the $Z = 50$ and $Z = 82$ closed-shell mass region as well as in strongly deformed
nuclei, the experimental low-lying extra states have been observed, that cannot be accounted for within the \((sd)^8\) boson space. It goes without saying that the experimental evidence for the importance of the hexadecapole degree of freedom has resulted. Another evidence such as \(1 - k^2 = 0^+, 3^+, 2^+_1, 2^+_2\), and \(4^+\) bands which can be interpreted as bands built on hexadecapole vibrations \(2 - \) hexadecapole transition densities for exciting some of the \(4^+\) levels in Cd, Pd, Os and Pt isotopes \(3 - \) the \(\text{B(E2)}\) values fall off too rapidly due to the boson cut-off \(4 - \) some calculated transitions are forbidden, while this contradicts the experimental data which can be occurred by introducing other degree of freedom. On the other hand, one of the main problems is that IBM could not provide a description for the triaxial nuclei. Investigation done in [37] for shapes in the sdg model indicated that \(1 - \) quadrupole interactions always lead to an axial shape, \(2 - \) a triaxial shape can be induced by adding a hexadecapole interaction. Thus sdg model could provide a better explanation for this purpose. Some other evidence show the necessity for inclusion of another degree of freedom in the IBA. One of them is the hexadecapole degree of freedom (g-boson) and one of the best models for this purpose is the \(sdg\) interacting boson model [32-41] in which the nuclear collective excitations of even–even nuclei are contains \(L = 0\) s-boson, \(L = 2\) d-boson and \(L = 4\) g-bosons nucleon pairs. Since the total single boson space is 15-dimensional, the symmetry group is \(U(15)\) possessing strong coupling dynamical symmetries \(SU(3), SU(5), SU(6)\), \(SO(15)\) and weak coupling dynamical symmetries \(U_{sd}(6) \otimes U_g(9), U_{dg}(14) \otimes U_s(1), U_d(5) \otimes U_{dg}(10).\) However, \(U_d(5) \otimes U_g(9), SU(3)\) and \(SO(15)\) are the analogous of the limits of the \(sd\)-IBM. The results [40,41] clearly indicate that \(SO(15)\) and \(SO(6)\) can be described \(\gamma\)-unstable nuclei. Also, \(U_d(5) \otimes U_g(9)\) and \(U(5)\) describe vibrational nuclei. So \(U_d(5) \otimes U_g(9)-SO(15)\) and \(SU(5)-SO(6)\) are vibrational to \(\gamma\)-unstable transitional region. The Hamiltonian of \(sdg\) model contains too many parameters (35 free parameters in total: three single particle energies and 32 two body matrix elements), so it is a complicated Hamiltonian which has a large basis space. Different approximations were applied to solve the large basis problem [36–39]. In the case of the transitional nuclei, the complicated numerical methods need to diagonalize the transitional Hamiltonian but Pan and Draayer [5] have proposed a new solution based on affine-\(SU(1, 1)\) algebraic technique and explores the properties of nuclei in the \(U(5)-SO(6)\) transitional region of IBM corresponding to vibrational to \(\gamma\)-unstable transition. Also, we attempt to investigate the transition \(U_d(5) \otimes U_g(9)-SO(15)\) in \(46^{104-110}\) Pd isotopes, which also, correspond to vibrational to \(\gamma\)-unstable transition. So, an exactly solvable method is suggested to solve the problem easily.

In this study, we investigate the transition of \(U_d(5) \otimes U_g(9)-SO(15)\) in \(104^{104-110}\) Pd isotopes, and introduced the new algebraic solution for these nuclei. In this method, the \(sdg\) Hamiltonian which is a three level pairing Hamiltonian is determined via the exactly solvable method. An approach related to Richardson’s formalism which can be reduced to the quasiparticle formalism in the appropriate limit is used. In fact, algebraic Bethe ansatz technique to the nuclear pairing problem is applied. We evaluate exact solutions for eigenstate and energy eigenvalues for the transitional region in the \(sdg\)IBM by using the dual algebraic structure for the three level pairing model and by changing the control parameter that based on affine \(SU(1, 1)\) Lie algebra. In order to investigate the phase transition, we calculate some observables such as energy levels, \(\text{BE}(2), \text{BE}(4)\) and expectation values of the g-boson number operator. The low-lying states of \(104^{104-110}\) Pd isotopes have been studied within the suggested model. The results of calculations for these nuclei will be presented for the Energy levels, transitions probabilities, two neutron separation energies, signature splitting of the \(\gamma\)-vibrational band and will be compared with the corresponding experimental data.

This paper is organized as follows: Section 2 briefly summarizes \(sdg\) model Hamiltonian. Section 3 includes theoretical aspects of transitional Hamiltonian and affine \(SU(1, 1)\) algebraic
technique. In sections 4, 5, some observables, the results and experimental evidence are discussed and sect. 6 is devoted to the summary and some conclusions.

2. \(\text{sdg}\) model Hamiltonians for transitional nuclei

Ref. [40] is the best review of \(\text{sdg}\)-IBM. The \(\text{sdg}\) interacting boson model in which nuclear collective excitations of even–even nuclei containing \(L = 0\) s-boson, \(L = 2\) d-boson and \(L = 4\) g-bosons nucleon pairs conserve the total number of bosons and since the total single boson space is 15 dimensional, the symmetry group is \(U(15)\) which is generated by 225 operators where \(b_{lm}^+(b_{lm})\) creates/annihilate a boson with angular momentum \(l\) and \(z\) projection \(m\). A Hamiltonian [40,41] that conserves the total number of bosons contains too many parameters, 35 parameters are required to determine the Hamiltonian of the \(\text{sdg}\)-IBM: three single-boson energies (\(\epsilon_s\), \(\epsilon_d\), \(\epsilon_g\)) and 32 two body matrix elements originating from this rule that the bosons are symmetrically coupled. As a results, the allowed two-boson states are \(s^2\) (\(L = 0\)), \(d^2\) (\(L = 0, 2, 4\)), \(g^2\) (\(L = 0, 2, 4, 6, 8\)), \(sd\) (\(L = 2\)), \(sg\) (\(L = 4\)), \(dg\) (\(L = 2, 3, 4, 5, 6\)). On the other hand, for \(n\) states with a given angular momentum one has \(n(n+1)/2\) interactions. So, 32 independent two-body interactions, \(v\), as defined in Eq. (4) are made: 6 for \(L = 0\), 10 for \(L = 2\), 1 for \(L = 3\), 10 for \(L = 4\), 1 for \(L = 5\), 3 for \(L = 6\) and 1 for \(L = 8\). The Hamiltonian of the \(\text{sdg}\)-IBM can be written as:

\[
\hat{H} = \hat{H}^{(1)} + \hat{H}^{(2)}
\]

The index refers to the order of the interaction in the generators of \(U(15)\). \(\hat{H}^{(1)}\) and \(\hat{H}^{(2)}\) can be written as:

\[
\hat{H}^{(1)} = \epsilon_s [s^+ \times \bar{s}]^{(0)} + \epsilon_d \sqrt{5}[d^+ \times \bar{d}]^{(0)} + \epsilon_g \sqrt{9}[g^+ \times \bar{g}]^{(0)} = \epsilon_s \hat{n}_s + \epsilon_d \hat{n}_d + \epsilon_g \hat{n}_g
\]

And

\[
\hat{H}^{(2)} = \sum_{l_1 \leq l_2, l_3 \leq l_4, L} \hat{v}_{l_1 l_2 l_3 l_4}^{L} [(b_{l_1}^+ \times b_{l_2}^+)^L \times (b_{l_3}^+ \times b_{l_4}^+)]_{0}^{(0)}
\]

Where the coefficient \(\hat{v}\)

\[
\hat{v}_{l_1 l_2 l_3 l_4}^{L} = \frac{2L + 1}{(1 + 2\delta_{l_1 l_2})(1 + 2\delta_{l_3 l_4})} \langle l_1 l_2; LM | \hat{H}_2 | l_3 l_4; LM \rangle
\]

Is related to the interaction matrix elements between the normalized two-boson states. As a result \(\hat{H}\) can be written as:

\[
\hat{H} = \epsilon_s \hat{n}_s + \epsilon_d \hat{n}_d + \epsilon_g \hat{n}_g + \sum_{l_1 \leq l_2, l_3 \leq l_4, L} \hat{v}_{l_1 l_2 l_3 l_4}^{L} [(b_{l_1}^+ \times b_{l_2}^+)^L \times (b_{l_3}^+ \times b_{l_4}^+)]_{0}^{(0)}
\]

These parameters are too many for applied usage. Furthermore, however, the algebraic solution of the \(\text{sdg}\) IBM is still lacking. So far, for IBM, calculating was performed by two-level algebraic model [6,7]. In this work, three level pairing model based on dual algebraic model is proposed for performing the Hamiltonian of the extension of interacting boson model (\(\text{sdg}\)IBM). We showed that, by using affine \(su(1,1)\) algebra, \(\text{sdg}\) Hamiltonian can be explained, providing a better interpretation of observables which cannot be explained in \(\text{sdg}\) IBM model. By proposed dual algebraic structure for the three level pairing model, exact energy and eigenstate of the \(\text{sdg}\) interacting boson model could be described. These results are used to calculate energy levels,
electric quadrupole transition probabilities, BE(2), electric hexadecapole transitions probabilities, BE(4), the two neutron separation energies and signature splitting of the \( \gamma \)-vibrational band in \({}^{104\text{–}110}\text{Pd}\). In the following sections, we clarify how possible simplifications create vision with exactly solvable models.

3. **Theoretical method based on exactly solvable model for transitional region**

3.1. **Exactly solvable Hamiltonian for \( U_d(5) \otimes U_g(9)\)-SO(15) transitional Hamiltonian**

3.1.1. **Pairing Hamiltonian**

It is known that pairing has a significant role in understanding the physics of the quantum many-body problems. Also, in understanding the physics of nuclear structure and is an important interaction in nuclei. The concept of pairing interaction was first reported by Racah within the context of a seniority coupling scheme [42] and continued for realistic nuclear systems [43] by following [44]. The nuclear pairing Hamiltonian [13–16], was solved by integrable models called Richardson–Gaudin (RG) models are derived from adding the Richardson exact solution [17] to the integrable model proposed by Gaudin [18]. Richardson exact solution is an algebraic approach to the pairing problem and rather a complex formalism which was not widely used in nuclear physics. Indeed, the quasispin formalism of Kerman [9] was applied to a limited class of seniority-conserving pairing interactions. Comprehensive studies of these interactions have been the focus of recent works based on the algebraic Bethe ansatz and infinite-dimensional Lie algebraic methods [4–9].

\[
H = \sum_{km} \epsilon_k b_{km}^+ b_{km} + \frac{1}{4} \sum_{kkmm} G_{kk} (-)^{j_k - \hat{m}} b_{k,-\hat{m}}^+ b_{k,-\hat{m}} (-)^{j_k - \hat{m}} b_{k,-m} b_{k,m}
\]  

(6)

Where \( k = 1, 2, \ldots, j_k = \text{integer} \) (bosons) and \( j_k = \text{half integer} \) (fermion). On the other hand, classification scheme for pairing models, and determining the eigenvalues and eigenvectors of \( H \) are significant. Duality relations provide a simple method for solving these problems.

There has been growing evidence for the existence of different types of duality relations [1–9] in theoretical nuclear physics. The existence of duality relations has proven to be a powerful tool in relating the Hamiltonians of number-nonconserving quasispin and the number-conserving unitary algebras for system with pairing interactions. In this article, an approach related to Richardson’s formalism which can be reduced to the quasispin formalism in the appropriate limit is used. In fact, algebraic Bethe ansatz technique to the nuclear pairing problem is applied.

3.2. **Dual algebraic structure**

Realizations of Lie algebras in terms of boson operators are of great interest for applications in a variety of problems in physics, most notably to oscillator problems in quantum mechanics and in algebraic models of rotation–vibration spectra of molecules (vibron model) and nuclei (interaction boson model).

By taking bilinear products of creation and/or annihilation operators \( b_m^+ b_m^+; b_m^+ b_m, b_m b_m \) [19, 26,46] the fundamental Lie algebra, Sp(2n, R) for boson, characterizing transformations of a many-boson system can be constructed. \( \hat{m}, m = 1, 2, \ldots \) range over the single-particle states of the system.

The algebra, Sp(2n, R) for boson, has two commuting subalgebras:
1) The number-conserving subalgebra $U(n)$.
2) The number-non conserving subalgebra $SU(1, 1)$ as studied in the following sections.

1) The number-conserving algebra $U(n)$:
The unitary algebra $U(n)$ can be constructed by taking bilinear products of creation and annihilation operators (spanned by the elementary one-body operators $b^+_mb_m$). The $U(n)$ algebra contains a subalgebra $SO(n)$ for the bosonic case. On the other hand, in nuclear physics, rotationally-invariant problems are considered, which for bosonic system algebra, should be terminated to physical $SO(3)$ angular momentum algebra. So, for such rotationally-invariant systems, the $SO(n)$ subalgebras in turn contain the physical $SO(3)$ angular momentum algebra. For such rotationally invariant systems labeled by $j$, it is suitable to use $m = j, j - 1, ..., -j$, and time-reversal creation and annihilation operators $b^+_m \rightarrow b^+_j$ and $b^+_m \rightarrow (-)^{j-m}b^+_j$. For a single value of $j$, the number of allowed states is $n = 2j + 1$. The representations of $U(n)$, characterized by the total number of particles $N$, are totally symmetric for bosons. The representations of $SO(n)$ contained in $U(n)$ are also totally symmetric (bosons). They are characterized by the quantum number $\nu$, called seniority, and introduced by Racah in 1949 [45] and is given by $\nu = (N\text{mod}2), ..., N - 2, N$. As a result, the classification scheme is:

$$\text{Sp}(2n, R) \supset U(n) \supset \left\{ SO(n) \supset \ldots \supset SO(3) \right\}$$

On the other hand, we should know that, if each single-particle creation operator $b^+_m$ is associated with a time-reversed partner $b^+_m$, then $SO(n)$ defined by the property will leave invariant the scalar pair state $\sum_m b^+_m b^+_m |0\rangle$ [46]. This special property underlies the duality relations with the quasispin pair algebra considered in this work.

(2) Number-nonconserving:
This is the so-called quasi-spin algebra composed of three elements

$$S^+ = \frac{1}{2} \sum_m b^+_m b^+_m, \quad S^- = \frac{1}{2} \sum_m b^+_m b^-_m, \quad S^z = \frac{1}{4} \sum_m (b^+_m b^-_m + \vartheta b^-_m b^+_m)$$

(8)

Where $\vartheta = +1$ for $SU(1, 1)$ (bosons). Let $|K\mu\rangle$ denote basis states of an irreducible representation (irrep) of $SU(1, 1)$, where $K$ can be any positive real number and $\mu = K, K + 1, ...$. Then representations of $SU(1, 1) \supset U(1)$ are characterized by $K$ (quasi-spin) and $\mu$ (quasi-spin projection). This gives the alternate decomposition.

$$\text{Sp}(2n, R) \supset SU(1, 1) \supset U(1)$$

(9)

States of pairing models can thus be classified in terms of two commuting algebras number-conserving algebra and number-nonconserving algebra with labels in one-to-one correspondence (duality relations)

$$K = \frac{1}{2} (\Omega + \vartheta \nu), \quad \mu = \frac{1}{2} (N + \vartheta \Omega)$$

(10)

Where $\Omega = (j + \frac{1}{2}) = \frac{N}{2}$. For a complete classification scheme of many-body bosonic systems, the algebras $SO(n)$ needs to be further reduced

$$\text{Sp}(2n, R) \supset \left\{ U(n) \supset SO(n) \supset \ldots \supset SO(3) \right\}$$

(11)

So, by the use of these algebras and subalgebras, we can define dual algebraic structure for sdg model and our transitional region.
3.3. Dual algebraic structure for the three level pairing model

As mentioned above, the quasispin and orthogonal algebras are subalgebras of the larger Sp(2n, R) algebra (as shown in Eq. (11)). So, a relation between unitary chain and quasispin subalgebras occur and they are mutually commuting “dual” algebras of Sp(2n, R) algebra. The sdg interacting boson model is a three level pairing model, for the three-level models the dual algebras are: 1) The number-conserving subalgebra U(n1 + n2 + n3). 2) The number-nonconserving subalgebras SU1(1, 1) ⊗ SU2(1, 1) ⊗ SU3(1, 1) [2,3,6,12]. So, similar to the issues already mentioned. We have

\[ Sp(2n, R) \triangleright \left\{ U(n_1 + n_2 + n_3) \right\} \]

\[ \quad \begin{cases} \quad SU_1(1, 1) \otimes SU_2(1, 1) \otimes SU_3(1, 1) \end{cases} \]

(12)

In the following section, the number-conserving algebra U(n1 + n2 + n3) and its subalgebras and also the number-nonconserving algebra SU1(1, 1) ⊗ SU2(1, 1) ⊗ SU3(1, 1) and the related items for three level pairing algebra will be mentioned.

3.4. Number-conserving algebra U(n1 + n2 + n3)

According to the duality relation, we need to know about the number-conserving algebra. Thus, U(n1 + n2 + n3) denoting that the pairing problem is only important for the two of subalgebras chains is introduced. These two subalgebra are

\[ U(n_1 + n_2 + n_3) \triangleright \left\{ \quad SO(n_1 + n_2 + n_3) \quad \right\} \]

\[ \quad \begin{cases} \quad U(n_1) \otimes U(n_2) \otimes U(n_3) \end{cases} \]

(13)

where n1 = 2j1 + 1, n2 = 2j2 + 1 and n3 = 2j3 + 1. As mentioned above, sdg interacting boson model in which the nuclear collective excitations of even–even nuclei contain L = 0 s-boson, L = 2 d-boson and L = 4 g-bosons nucleon pairs is three level pairing model. Besides, the pairing problem is only important for the two of subalgebras chains (strong coupling and weak coupling shown in Eq. (13)). In the domain of nuclear physics, these two subalgebras are related to the vibrational–rotational limits (e.g.; vibrational to γ-unstable transition), and further studies have been done in the domain of these two subalgebras. Since the total single boson space in sdg IBM is 15 dimensional, the symmetry group is U(15) possessing strong coupling dynamical symmetries SU(3), SU(5), SU(6), SO(15) and weak coupling dynamical symmetries U_{sdg}(6) ⊗ \text{U}_g(9), U_{sdg}(14) ⊗ U_s(1), U_d(5) ⊗ U_{sdg}(10). However, U_d(5) ⊗ U_g(9), SU(3) and SO(15) are the analogous of the limits of the sd-IBM. The results clearly indicate that [40,41] SO(15) and SO(6) can describe γ-unstable nuclei. Also, U_d(5) ⊗ U_g(9) and U(5) describe vibrational nuclei. So, U_d(5) ⊗ U_g(9)-SO(15) and U(5)-SO(6) are vibrational to γ-unstable transition.

So, by using this method for U(15) (where n1 = 1, n2 = 5, n3 = 9 and n1 + n2 + n3 = 15, we introduced

\[ U(15) \triangleright \left\{ \quad SO(15) \right\} \]

\[ \quad \begin{cases} \quad U(1) \otimes U(5) \otimes U(9) \end{cases} \]

\[ \quad \begin{cases} \quad SO(1) \otimes SO(5) \otimes SO(9) \end{cases} \]

\[ \quad \begin{cases} \quad SO_3(3) \otimes SO_9(3) \end{cases} \]

\[ \quad \begin{cases} \quad SO_{dg}(3) \end{cases} \]

(14)

The branching rules of the reductions involved and generators of each subgroups are given in Table 1.
3.5. Number-nonconserving algebra $SU_1(1, 1) \otimes SU_2(1, 1) \otimes SU_3(1, 1)$

The subalgebra of three level quasi-spin algebra is:

$$SU_1(1, 1) \otimes SU_2(1, 1) \otimes SU_3(1, 1) \supset \left\{ SU_{123}(1, 1), U_1(1) \otimes U_2(1) \otimes U_3(1) \right\} \supset U_{123}(1)$$ (15)

As a result, for $sdg$ IBM, this chain have:

$$SU_s(1, 1) \otimes SU_d(1, 1) \otimes SU_g(1, 1) \supset \left\{ SU_{sdg}(1, 1), U_s(1) \otimes U_d(1) \otimes U_g(1) \right\} \supset U_{sdg}(1)$$ (16)

3.6. Application of affine $SU(1, 1)$ algebra for $U_d(5) \otimes U_g(9)$-SO(15) transitional Hamiltonian based on dual algebraic

Quasi-spin algebra contributes significantly to our investigation into this field as explained in detail in [5]. By using this relation, we can explain our method which can be used for $sdg$ IBM. In $sdg$IBM, the generators of $SU^g(1, 1)$, $SU^d(1, 1)$ and $SU^s(1, 1)$ respectively are generated by the $g$-boson pairing algebra, $d$-boson pairing algebra and $s$-boson pairing algebra

$$S^+(g) = \frac{1}{2}(g^+ \cdot g^+), \quad S^-(g) = \frac{1}{2}(g^{-} \cdot g^{-}), \quad S^0(g) = \frac{1}{2}(\vec{n}_g + \frac{9}{2})$$

$$S^+(d) = \frac{1}{2}(d^+ \cdot d^+), \quad S^-(d) = \frac{1}{2}(\vec{d} \cdot \vec{d}), \quad S^0(d) = \frac{1}{2}(\vec{n}_d + \frac{5}{2})$$

$$S^+(s) = \frac{1}{2}(s^+ \cdot s^+), \quad S^-(s) = \frac{1}{2}(\vec{s} \cdot \vec{s}), \quad S^0(s) = \frac{1}{2}(\vec{n}_s + \frac{1}{2})$$ (17)
And SU\textsubscript{sdg}(1, 1) is the s, d and g boson pairing algebras generated by

\begin{equation}
S^+(sdg) = \frac{1}{2}(g^+ \cdot g^+ + d^+ \cdot d^+ + s^+ s^+), \\
S^-(sdg) = \frac{1}{2}(g \cdot \bar{g} + \bar{d} \cdot d + s^2), \\
S^0(sdg) = \frac{1}{2}(\hat{n}_g + \frac{9}{2}) + \frac{1}{2}(\hat{n}_d + \frac{5}{2}) + \frac{1}{2}(\hat{n}_s + \frac{1}{2})
\end{equation}

Because of the duality relationships, it is known that the base of

\begin{equation}
U_d(5) \otimes U_g(9) \supset SO_d(5) \otimes SO_g(9) \supset SO_d(3) \otimes SO_g(3) \supset SO_{sdg}(3)
\end{equation}

and

\begin{equation}
SO_{sdg}(15) \supset SO_d(5) \otimes SO_g(9) \supset SO_d(3) \otimes SO_g(3) \supset SO_{sdg}(3)
\end{equation}

are simultaneously the basis of SU\textsubscript{d}(1, 1) \otimes SU\textsubscript{g}(1, 1) ⊃ U(1) and SU\textsubscript{sdg}(1, 1) ⊃ U(1), respectively. By the use of these relations, the Casimir operators of SO(5) \otimes SO(9) and SO(15) can also be expressed in terms of the Casimir operators of SU\textsubscript{d}(1, 1) \otimes SU\textsubscript{g}(1, 1), and SU\textsubscript{sdg}(1, 1), respectively.

\begin{equation}
\hat{C}_2(SU^d(1, 1)) = \frac{5}{16} + \frac{1}{4} \hat{C}_2(SO(5)) \\
\hat{C}_2(SU^g(1, 1)) = \frac{45}{16} + \frac{1}{4} \hat{C}_2(SO(9)) \\
\hat{C}_2(SU^{sdg}(1, 1)) = \frac{165}{4} + \frac{1}{4} \hat{C}_2(SO(15))
\end{equation}

Moreover the infinite dimensional SU(1, 1) algebra is generated by the use of

\begin{equation}
S^\pm_n = c_s^{2n+1} S^\pm(s) + c_d^{2n+1} S^\pm(d) + c_g^{2n+1} S^\pm(g), \quad S^0_n = c_s^{2n} S^0(s) + c_d^{2n} S^0(d) + c_g^{2n} S^0(g)
\end{equation}

Where c\textsubscript{s}, c\textsubscript{d} and c\textsubscript{g} are the real parameters and n can be 0, ±1, ±2, .... These generators satisfy the commutation relations \([S^0_m, S^\pm_n] = \pm S^\pm_{m+n}, [S^+_m, S^-_n] = -2S^0_{m+n+1}\). By employing the generators of Algebra SU(1, 1), the proposed Hamiltonian for transitional region between U\textsubscript{d}(5) \otimes U\textsubscript{g}(9)-SO(15) (vibrational to \gamma-unstable transition) limits is

\begin{equation}
\hat{H} = S^0_0 + \alpha S^0_0 + \gamma \hat{C}_2(SO_g(9)) + \delta \hat{C}_2(SO_d(5)) + + \chi \hat{C}_2(SO_d(3)) + \eta \hat{C}_2(SO_g(3)) + \beta \hat{C}_2(SO_{sdg}(3))
\end{equation}

Hamiltonian (21) is the suggested one for sdg systems and \(\alpha, \gamma, \delta, \chi, \eta, \beta\) are the real parameters. As shown in [41] for \(U_d(5) \otimes U_g(9)-SO(15)\) transition and by using the concepts in [5], it can be shown that, Hamiltonian (21) is equivalent to SO(15) Hamiltonian when \(c_s = c_d = c_g\) and with \(U_d(5) \otimes U_g(9)\) Hamiltonian if \(c_s = 0\) and \(c_d \neq c_g \neq 0\). Thus, the \(c_s \neq c_d \neq c_g \neq 0\) situation just corresponds to \(U_d(5) \otimes U_g(9)-SO(15)\) transitional region. In our calculation, we take \(c_d = 1\) constant value and have \(C_s = \frac{C_s}{c_d} \leq 1, C_g = \frac{C_g}{c_d} = 1\) (\(C_s\) varies between 0 and 1). \(\alpha, \gamma, \delta, \chi, \eta, \beta\) are fit parameters and \(C_s\) and \(C_g\) are control parameters, similar to the procedure in [5]. For evaluating the eigenvalues of Hamiltonian (21) the eigenstates are considered as
\[ |k; \nu_{vd}v_{s}n_{\Delta LM}\rangle = \sum_{n_{i} \in \mathbb{Z}} a_{n_{1}n_{2}...n_{k}}x_{1}^{n_{1}}x_{2}^{n_{2}}...x_{k}^{n_{k}}S_{n_{1}}^{+}S_{n_{2}}^{+}...S_{n_{k}}^{+}|lw\rangle \]  

(22)

Eigenstates of Hamiltonian (21) can obtain with using the Fourier–Laurent expansion of eigenstates and SU(1, 1) generators in terms of c-unknown number parameters \( x_{i} \) with \( i = 1, 2, ..., k \). It means, one can consider the eigenstates as

\[ |k; \nu_{vd}v_{s}n_{\Delta LM}\rangle = N S^{+}(x_{1})S^{+}(x_{2})...S^{+}(x_{k})|lw\rangle \]

(23)

Where \( N \) is the normalization factor and

\[ S_{x_{i}}^{+} = \frac{c_{s}}{1 - c_{s}^{2}x_{i}}S^{+}(s) + \frac{c_{d}}{1 - c_{d}^{2}x_{i}}S^{+}(d) + \frac{c_{g}}{1 - c_{g}^{2}x_{i}}S^{+}(g) \]

(24)

The c-numbers \( x_{i} \) are determined through the following set of equations:

\[ \frac{\alpha}{x_{i}} = \frac{gC_{s}^{2}(v_{s} + \frac{1}{2})}{1 - C_{s}^{2}x_{i}} + \frac{g(v_{d} + \frac{5}{2})}{1 - x_{i}} + \frac{gC_{g}^{2}(v_{g} + \frac{9}{2})}{1 - C_{g}^{2}x_{i}} - \sum_{j \neq i} \frac{2g}{x_{i} - x_{j}} \]

(25)

Let \( |lw\rangle \) be the lowest weight state of \( (SU(1, 1)) \), which should satisfy

\[ S^{-}(s)|lw\rangle = 0, \quad S^{-}(d)|lw\rangle = 0, \quad S^{-}(g)|lw\rangle = 0 \]

Since the total number of bosons, \( N \), is a conserved quantity, and the basis vectors are restricted under the sub algebra chain

\[ SO_{sdg}(15) \supset SO_{d}(5) \otimes SO_{g}(9) \supset SO_{d}(3) \otimes SO_{g}(3) \supset SO_{dg}(3), \]

the lowest weight states satisfying above equation \( (S^{-}(s)|lw\rangle = 0, S^{-}(d)|lw\rangle = 0, S^{-}(g)|lw\rangle = 0) \) are actually a set of basis vectors of

\[ U_{d}(5) \otimes U_{g}(9) \supset SO_{d}(5) \otimes SO_{g}(9) \supset SO_{d}(3) \otimes SO_{g}(3) \supset SO_{dg}(3) \]

with the lowest weight state, \( |lw\rangle \):

\[ |lw\rangle = |N, k_{g} = \frac{1}{2}(v_{g} + \frac{9}{2}), \mu_{g} = \frac{1}{2}(n_{g} + \frac{9}{2}), k_{d} = \frac{1}{2}(v_{d} + \frac{5}{2}), \mu_{d} = \frac{1}{2}(n_{d} + \frac{5}{2}), k_{s} = \frac{1}{2}(v_{s} + \frac{1}{2}), \mu_{s} = \frac{1}{2}(n_{s} + \frac{1}{2}), n_{\Delta}, L_{d}, L_{g}, L_{dg}, M_{dg}\rangle \]

where \( N = v_{s} + v_{d} + v_{g}, n_{s} = v_{s}, n_{d} = v_{d}, n_{g} = v_{g} \). Hence, we have

\[ S^{0}_{n}|lw\rangle = \Lambda^{0}_{n}|lw\rangle \quad \Lambda^{0}_{n} = c_{s}^{2n}(v_{s} + \frac{1}{2})^{2} + c_{d}^{2n}(v_{d} + \frac{5}{2})^{2} + c_{g}^{2n}(v_{g} + \frac{9}{2})^{2} \]

(26)

Then by using these relations, the eigenvalues of Hamiltonian (21) can be expressed:

\[ E^{(k)} = \hbar^{(k)} + \alpha \Lambda^{0}_{1} + \gamma \left(v_{g}(v_{g} + 7)\right) + \delta \left(v_{d}(v_{d} + 3)\right)
+ \chi \left(L_{d}(L_{d} + 1)\right) + \eta \left(L_{g}(L_{g} + 1)\right) + \beta \left(L(L + 1)\right) \]

(27)

Where \( \hbar^{(k)} = \sum_{i=1}^{k} \frac{\alpha}{x_{i}} \) and the quantum number \( (k) \) is related to the total boson number \( N \) by \( N = 2k + v_{s} + v_{d} + v_{g} \). In order to obtain the numerical results for energy spectra of the considered nuclei, a set of non-linear Betheansatz equations (BAE) with \( k \)-unknowns for \( k \)-pair excitations must be solved. Also the constants of Hamiltonian with the least square fitting processes to the experimental data are obtained. To achieve this aim, we have changed variables as
\[
C_s = \frac{C_s}{c_d} \leq 1 , \quad C_g = \frac{C_g}{c_d} = 1 , \quad y_i = c_d^2 x_i \\
\alpha = \frac{C_s^2 (v_s + \frac{1}{2})}{1 - C_s^2 y_i} + \frac{(v_d + \frac{5}{2})}{1 - y_i} + \frac{(v_g + \frac{3}{2})}{1 - y_i} - \sum_{j \neq i} \frac{2}{y_i - y_j} 
\]

To calculate the roots of Bethe–Ansatz equations (BAE) with specified values of \(v_s, v_d\) and \(v_g\), we have solved Eq. (28) with definite values of \(C_s\) and \(\alpha\). Then, we carry out this procedure with different values of \(C_s\) and \(\alpha\) to give energy spectra with minimum variation in compare to experimental values [47];

\[
\sigma = \left( \frac{1}{N_{\text{tot}}} \sum_{i=1}^{N_{\text{tot}}} |E_{\text{exp}}(i) - E_{\text{calc}}(i)|^2 \right)^{\frac{1}{2}}
\]

\((N_{\text{tot}}\) the number of energy levels where included in the fitting processes). The method for optimizing the set of parameters in the Hamiltonian \(\gamma, \delta, \chi, \eta, \beta\) includes carrying out a least-square fit (LSF) of the excitation energies of selected states.

4. Observables

The significant changes in the energy levels, electromagnetic transition rates and expectation values of the g-boson number operator resulting in the shape phase transitions from one kind of collective behavior to another have been employed as the worthy observables to analyze the phase transition. The method for achieving the energy levels was discussed in the previous section and in the following, we discuss other observables.

4.1. \(B(E2)\) and \(B(E4)\) transitions

The electromagnetic transitions such as electric quadrupole transition probabilities, \(B\) (E2), and electric hexadecupole transition probabilities, \(B\) (E4), are worthy observables that provide important information about QPTs, have been an important test for collective models of nuclei. In this section, we discuss the calculation of E2 transition strengths. Explicit form for electric quadrupole \(T\) (E2) is

\[
T_{\mu}^{(E2)} = q_{02}^{(2)} [s^+ \times \tilde{d} + d^+ \times \tilde{s}]_{\mu}^{(2)} + q_{22}^{(2)} [d^+ \times \tilde{d}]_{\mu}^{(2)} + q_{24}^{(2)} [d^+ \times \tilde{g} + g^+ \times \tilde{d}]_{\mu}^{(2)} + q_{44}^{(2)} [g^+ \times \tilde{g}]_{\mu}^{(2)}
\]

Where, \(q_{--}\)'s is the effective charge [40] express in \(\sqrt{W.u}\) unit. By using selection rule, \(\text{BE}(2)\) is given by

\[
B(E2; \alpha_i J_i \rightarrow \alpha_f J_f) = \frac{|\langle \alpha_f J_f | T^{(E2)} | \alpha_i J_i \rangle|^2}{2 J_f + 1}
\]

By following the procedure in Ref. [29], which use CFP, the numerical calculation of the reduced matrix elements can be carried on. However, we are interested in obtaining analytic expressions for these matrix elements and we will therefore describe in this section an alternative approach to obtain matrix elements in algebraic form. According to Refs. [5,10,29]. We calculate the matrix elements among states of maximum seniority. For evaluating matrix elements, we consider eigenstates (22) which consist of \(|lw\), the lowest weight state of \((SU(1, 1))\). The quantum number in the \(|lw\) take some special value. Indeed, for low state, we have
N = v_s + v_d + v_g, n_s = v_s, n_d = v_d, n_g = v_g, which obtained based on N = 2k + v_s + v_d + v_g. When the ground state which does not have any pair is considered. By using these special value (n_s = v_s, n_d = v_d, n_g = v_g), Wigner–Eckart theorem, bosonic commutation relations and Clebsch–Gordon coefficient, we calculate the element of reduced matrix element and B(E2) also to determine the q−...’s we have followed the procedure in Refs. [5,10]. For evaluating B(E2), we consider eigenstates (22) from which the normalization factor is obtained as

\[
\mathcal{N} = \sqrt{\frac{1}{\prod_{p=1}^{k} \sum_{i=p}^{k} \frac{2C_2^k(k-p+\frac{1}{2})(\nu_s+\frac{1}{2})}{(1-C_2^k\nu_{k+1-p})(1-C_2^k\nu_{i})} + \frac{2(k-p+\frac{1}{2})(\nu_d+\frac{1}{2})}{(1-C_2^k\nu_{k+1-p})(1-C_2^k\nu_{i})} + \frac{2C_2^k(k-p+\frac{1}{2})(\nu_g+\frac{1}{2})}{(1-C_2^k\nu_{k+1-p})(1-C_2^k\nu_{i})}}}.
\]

(31)

Another observable is electric hexadecupole \( T(E4) \)

\[
T^{(E4)}_\mu = q_{04}^{(4)}[s^+ \times \tilde{g} + g^+ \times \tilde{s}]^{(4)}_\mu + q_{22}^{(4)}[d^+ \times \tilde{d}]^{(4)}_\mu + q_{24}^{(4)}[d^+ \times \tilde{g} + g^+ \times \tilde{d}]^{(4)}_\mu + q_{44}^{(4)}[s^+ \times \tilde{g}]^{(4)}_\mu.
\]

(32)

Where, q−...’s is the effective charge [40]. Like \( BE(2) \), \( BE(4) \) can be calculated by using selection rule, but in this transition \( k = 4 \). Also like \( BE(2) \) by considering eigenstates (22), the normalization factor is obtained. Some large E4 matrix element as well as the forbidden one, could be explained by \( BE(4) \) operator presented in Eq. (32).

4.2. Expectation values of the g-boson number operator

An appropriate quantal order parameter is:

\[
\langle \hat{n}_g \rangle = \frac{\langle \psi | \hat{n}_g | \psi \rangle}{\mathcal{N}}.
\]

In order to obtain \( \langle \hat{n}_g \rangle \), we act \( s^0_m \) on the eigenstate, \(|k; \nu_g \nu_d \nu_s n_{\Delta} LM\rangle\)

\[
\langle \hat{n}_g \rangle = \frac{2(1 + c^2_f)(\Lambda^0_0 - \Lambda^0_2 + k(1 - \sum_{i=1}^{k} x_i^{-2}) + 2(\Lambda^0_0 - \Lambda^2_0 + k(1 - \sum_{i=1}^{k} x_i^{-2}))}{(1 - c^2_g)(c^2_f - c^2_g)(N)} - \frac{9}{2N}
\]

(33)

where

\[
\Lambda^0_\mu = c^2_f (\nu_s + \frac{1}{2}) + c^2_d (\nu_d + \frac{5}{2}) + c^2_g (\nu_g + \frac{1}{2})
\]

By using such relation, calculations for \( ^{104-110}\text{Pd} \) isotopes can be easily explained. This quantal order parameter is discussed in the following section.

5. Application to selected transitional nuclei \( ^{104-110}\text{Pd} \) and comparison with theoretical data

This section presents the calculated results of the low-lying states of \( ^{104-110}\text{Pd} \). The results include energy levels, B(E2), BE(4), the two neutron separation energies, signature splitting of the \( \gamma \)-vibrational band and expectation values of the g-boson number operator which are used for investigation of transitional characteristics of even–even Pd isotopes which intermediate between spherical and gamma-unstable shapes.
5.1. Energy spectrum

The nuclei in the mass regions around $A \sim 100$ [48,49] and $A \sim 130$ [50] have transitional characteristics intermediate between spherical and $\gamma$-unstable shapes. The theoretical and experimental studies of energy spectra done in [51] show the pd isotopes have spherical to gamma-unstable transitional characteristics. These nuclei have been extensively studied [48,49,51–55] because of transitional behavior of their level structures and other quantities. In [56], by means of the inelastic electron scattering, the low-lying excited states of $^{104–110}$Pd isotopes have been investigated and energy spectra and transition strengths such as BE(2) and BE(4) have been computed. Existence of the hexadecapole strength indicated the need for including g-bosons in the IBM to reproduce the $E4$ data. Additionally, the $sdg$ model could provide a good description of the observed $BE4$ data. On the other hand, in many transitional nuclei near the closed shells such as Zr, Mo, Ru, Pd, Cd, Sn adjacent to the $Z = 50$ and $Z = 82$ closed-shell mass region as well as in strongly deformed nuclei, the experimental low-lying extra $0^{+}$, $2^{+}$, $4^{+}$ states have been observed, that cannot be accounted for within the $(sd)^{N}$ boson space. Also, as previously mentioned, some band such as $k^2 = 0^{+}_3$, $3^{+}_1$, $2^{+}_2$, and $4^{+}$ bands can be interpreted as the bands built on hexadecapole vibrations. One of the most complicated form of transitional nuclei is the triaxial nature of their energy surface which is neither $\gamma$-unstable as in the Wilets–Jean model [57] nor $\gamma$-rigid as in the Davydov–Filippov model [58] but rather $\gamma$-soft which introduced by collective model of nuclei years ago [59]. And, it is still a subject of investigations. Investigation done in [37] for shapes in the $sdg$ model indicated that $1$ – quadrupole interactions always lead to an axial shape, $2$ – a triaxial shape can be induced by adding a hexadecapole interaction. Thus $sdg$ model could provide a better explanation for this purpose. Furthermore, the hexadecapole strength indicated the need for including g-bosons in the IBM to reproduce the $E4$ data. Consequently, the $sdg$ model could provide better explanation for the observed $BE4$ data. So, by following the theoretical method achieved beforehand, we next apply our algebraic model for $sdg$ model to the $^{104–110}$Pd isotopes.

IBM is designed for the low-lying states, usually with energies less than 2.5 MeV. But with introducing g boson, investigation of higher state is possible and the level is described better. For example, in $sd$ calculation, $6^{+}_1$ state has $n_d = 3$, however in the $sdg$ framework, by introducing g boson, the $6^{+}_1$ state has the structure $n_d = 1$, $n_g = 1$. So, we investigate these states with our model to achieve good results. Also higher state like $0^{+}_4$, $2^{+}_4$, $4^{+}$ (g-boson state), $4^{+}_4$, $5^{+}_1$, $6^{+}_2$,.. and $10^{+}_2$ are investigated and good results is achieved. The calculated energy levels match very well with the experimental levels. In this study, a simplifying assumption is made that we can therefore analyze the positive parity states of the, $^{104–110}$Pd. In order to obtain energy spectrum and realistic calculation for these isotopes, we need to specify Hamiltonian parameters (21). Eigenvalues of these systems are obtained by solving Bethe–Ansatz equations with least square fitting processes to experimental data to obtain constants of Hamiltonian. The advantage of the Bethe Ansatz wave function is that one can gain some physical insight from the rapidities $\xi_1$ in Eq. (22), as it is known that the structure of the rapidities can shed light on the collectivity of the boson pairs, such as is the case in the fermionic case [60]. For getting the best fit, six parameters of the suggested Hamiltonian have been carefully adjusted to the experimental energies of $^{104–110}$Pd (each Pd isotope). In our calculation, we take $c_{d}(=1)$ constant value and have $C_{s} = \frac{c_{g}}{c_{d}} \leq 1, C_{g} = \frac{c_{s}}{c_{d}} = 1$ ($C_s$ varies between 0 and 1). $\alpha$, $\gamma$, $\delta$, $\chi$, $\eta$, $\beta$ are fit parameters and $C_s$ and $C_g$ are control parameters, similar to the procedure in Ref. [5]. The best set of tran-
Table 2
(a) Parameters of Hamiltonian (21) used in the calculation of the $^{104-110}$Pd Isotopes and $c_s = 0.63, 0.69, 0.75, 0.80 (c_s = 1)$ respectively. All parameters are given in keV. (b) Parameters of sd-IBM Hamiltonian and rms deviations used in the calculation of the $^{104-110}$Pd Isotopes and $c_s = 0.63, 0.69, 0.75, 0.80$ respectively. All parameters are given in keV.

(a)

<table>
<thead>
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<th>Nucleus</th>
<th>$N$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\delta$</th>
<th>$\chi$</th>
<th>$\eta$</th>
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<tbody>
<tr>
<td>$^{104}$Pd</td>
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<td>65.12</td>
<td>16.34</td>
<td>−2.96</td>
<td>−3.07</td>
<td>−1.095</td>
<td>−0.181</td>
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</table>

(b)

<table>
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<th>Nucleus</th>
<th>$N$</th>
<th>$\alpha$</th>
<th>$\gamma$</th>
<th>$\delta$</th>
<th>$\sigma$</th>
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<td>373.6</td>
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</tbody>
</table>

transitional Hamiltonian parameters, i.e., $\alpha$, $\gamma$, $\delta$, $\chi$, $\eta$, $\beta$ have been achieved via least-square fit to the available experimental data \[47\] for excitation energies of the selected states $0^+_1$, $2^+_1$, $0^+_2$, $2^+_2$, $4^+_4$, $2^+_3$, $0^+_2$, $3^+_1$, $4^+_2$, $6^+_1$, $2^+_4$, $4^+_3$, $4^+_4$, $5^+_1$, $6^+_2$, $8^+_1$, $7^+_1$, $8^+_2$, $10^+_1$, $9^+_1$, $10^+_2$. To optimize the Hamiltonian parameters set, we then repeat this procedure, with different values of $c_s$ to minimize the root mean square (rms) deviation $\sigma$ between the calculated energy spectra and experimental counterparts which explore the quality of extraction processes. The deviation is defined by the equality $\sigma = (\frac{1}{N_{tot}} \sum_{i,tot} |E_{exp}(i) - E_{Cal}(i)|^2)^{1/2} (\text{keV})$ ($N_{tot}$ where $N_{tot}$ is the number of energy levels included in extraction procedure) the estimated results for Hamiltonian quantities which describe $^{104-110}$Pd isotopes. $\sigma$ is specified the quality of the fits. The final values of sdg-IBM fit’s parameters and control parameter, $c_s$, for $^{104-110}$Pd are given in Table 2(a). Also, according to the calculation done in Ref. \[5\], the energy spectra for sd-IBM are computed. Subsequently, we compare these results with sdg IBM. The final values of sd-IBM fit’s parameters, control parameter, $c_s$, and rms deviations which describe the quality of our estimation for $^{104-110}$Pd are given in Table 2(b). A comparison between sd-IBM and sdg-IBM theoretical prediction for the energy spectra and the experimental counterparts of Pd isotopes and rms deviations which describe the quality of our estimation for each isotope of $^{104-110}$Pd are plotted in Fig. 1. According to the calculations performed in the wake of sd IBM, we conclude that the states mentioned above are better described with sdg IBM. The sdg IBM calculations comparing to the sd IBM will have less $\sigma$ with better results. Our results proposed that isotopes of $^{104-110}$Pd are good example of transitional nuclei. Energy ratios such as $(E(4_1)/E(2_1))$ and $(E(0_2)/E(2_1))$ are the most basic structural predictions of spherical to $\gamma$-unstable shape-phase transition. For example, for $(E(4_1)/E(2_1))$, the ratio equal to 2.2–2.5 indicates the spectrum of transitional nuclei in the spherical to $\gamma$-unstable region \[20,21\]. Thus we calculated this quantity for Pd Isotopes. Fig. 2 show these prediction values for Pd Isotopes. It displays that $(E(4_1)/E(2_1))$ values for $^{104-110}$Pd isotopes are approximately 2.35–2.5.
Fig. 1. (Color online.) Comparison between experimental and calculated (sd-IBM and sdg-IBM) energies (in keV) for the positive parity states of even–even Pd Isotopes. The r.m.s. deviation values of the sdg-IBM calculation are 145.36, 132.32, 128.36, 139.32 for $^{104−110}$Pd, respectively and r.m.s. deviation values of the sd-IBM calculation are 324.3, 402.5, 415.1, 373.6 for $^{104−110}$Pd, respectively. The experimental data have been taken from [47].

Fig. 2. (Color online.) A comparison between theoretical (sdg-calculation) and experimental ($E(4_1)/E(2_1)$) and ($E(0_2)/E(2_1)$) prediction values for Pd Isotopes. The experimental data have been taken from [47].

5.2. $B(E2)$ and $B(E4)$ transitions

The reduced electric transition probabilities are considered as good observables within the low-lying state bands which prepare further information about the nuclear structure. Although the standard IBM-I with only $s$ and $d$ bosons can describe the $E2$ properties of pd isotopes quite well, a more generalized version which also includes a $g$-boson is needed to correctly reproduce
simultaneously both the E2 and E4 data. Some calculated transition in sd are forbidden, while the experimental data exists. For example in sd calculation, as said above a 6 state has \( n_d = 3 \), however by introducing g boson, the 6 state has structure \( n_d = 1, n_g = 1 \). In the fitting procedure, a large portion of the calculated transition is confirmed very well with the experimental values. Transition for higher states such as \( (2)^+_3 \rightarrow (0)^+_{2} \) and \( (0)^+_3 \rightarrow (2)^+_{2} \) and also for some \( 4^+_i \) states such as \( (4)^+_1 \rightarrow (2)^+_{1} \) is performed and the calculated transition is confirmed very well with the experimental values. Also, according to the calculation done in Ref. [5], the \( B(E2) \) for sd-IBM are computed. Subsequently, we compare these results with sdg IBM. There are four parameters in the fit of sdg IBM. We show that the E2 transitions can be explained with a simple algebraic method which can calculate all four parameters of this quantity. \( q_0^{(2)}, q_2^{(2)}, q_4^{(2)} \) and \( q_4^{(2)} \) refer to the \( s-d, d-d, d-g \) and \( g-g \) terms. The final values for this fit are listed in Table 3(a). There are two parameters in the fit of sd-IBM. We show that the E2 transitions can be explained with an algebraic method which can calculate all two parameters of this quantity. \( q_0^{(2)} \) and \( q_2^{(2)} \) refer to the \( s-d \) and \( d-d \) terms. The final values for this fit are listed in Table 3(b). Likewise, the results of the sdg and sd calculations of \( B(E2) \) can be seen in Table 4. According to the calculations performed for sd IBM, we conclude that the sdg IBM provides a better interpretation of transition for higher states such as \( (2)^+_3 \rightarrow (0)^+_{2} \) and \( (0)^+_3 \rightarrow (2)^+_{2} \) rather than the sd boson model. The sdg IBM calculations comparing to the sd IBM will have less \( \sigma \) with better results. Another structural predictions of spherical to \( \gamma \)-unstable shape-phase transition are \( B(E2) \) and \( B(E4) \) ratio show in Fig. 3.

The results of the calculated \( B(E4) \) are illustrated in Table 5. We calculate the E4 matrix elements which cannot be explained in the \( sd \) model. For this transition, we show that the E4 transitions can be explained with a simple algebraic method, is calculated all four parameters of this quantity. By using Eq. (32) which is given for E4 transition, \( B(E4) \) transition for some states like \( (4)^+_3 \rightarrow (0)^+_{1} \) and \( (4)^+_4 \rightarrow (0)^+_{1} \) is performed and the calculated transition is admitted very well with the experimental values. In the fitting procedure, calculated transition admitted
very well with the experimental values. There are four parameters in the fit, \( q_{04}^{(4)} \), \( q_{22}^{(4)} \), \( q_{24}^{(4)} \) and \( q_{44}^{(4)} \) referring to the \( s\)-\( g \), \( d\)-\( d \), \( d\)-\( g \) and \( g\)-\( g \) terms. The final values for this fit are listed in Table 6. Some of these transition are fitted in Fig. 4.

### 5.3. Investigation of triaxiality

It is well known that transitional nuclei have many interesting features such as odd–even staggering (OES) in the gamma band at low spins. As was shown in several studies \([20,21]\), these nuclei show clear triaxial feature. By triaxial features the fact that the \( \gamma \)-degree of freedom plays an important role in the description of these nuclei is realized. One of the most complicated form of transitional nuclei is the triaxial nature of their energy surface which is neither \( \gamma \)-unstable as in the Wilets–Jean model \([58]\) nor \( \gamma \)-rigid as in the Davydov–Filippov model \([59]\) but rather \( \gamma \)-soft introduced by the collective model of nuclei years ago \([60]\). It is worth mentioning that, it is still a subject of investigations. One of the main problems is that IBM could not provide a description for the triaxial nuclei which could be described by algebraic collective model (ACM) \([61,62]\), which is an algebraic version of the Bohr model. To describe triaxiality in the ACM a term proportional to \( \cos^2(3\gamma) \) must be included in the Hamiltonian. \( \cos^2(3\gamma) \) is proportional to the cubic terms. IBM consists of one and two body terms, excluding this term. So, by introducing the cubic terms in the Hamiltonian \([63–66]\) such term can be acquired. Such cubic term could be considered as an effective interaction resulting from the renormalization of higher angular momentum such as \( g \) \([67]\). However, this procedure is obscure and calculation directly in the sdg boson model would be favorable. Investigation done in \([37]\) for shapes in the sdg model indicated that 1 – quadrupole interactions always lead to an axial shape, 2 – a triaxial shape can be induced by adding a hexadecapole interaction. Thus sdg model could provide a better explanation for this purpose. The staggering is a signature of QPT in different categories of transitional regions into a list of cases: a) \( \gamma \)-soft, between a spherical vibrator and a deformed \( \gamma \)-soft rotor or \( \gamma \)-unstable structure corresponding to the U(5) to O(6) limits containing the critical point symmetry E(5); b) axially symmetric, for those between a spherical vibrator and a \( \gamma \)-rigid rotor corresponding to the U(5) to SU(3) limits containing the critical point symmetry X(5); and c) triaxial or \( \gamma \)-rigid, for nuclei between a spherical vibrator and a rigid triaxial rotor corresponds to fixed \( \gamma \) values between 0 and 30 but in our literature, energy staggering occurring between the U(5) and the

### Table 4

Comparison of \( BE2 \) from \(^{104–110}\text{Pd} \) with the \( sdg \) and the \( sd \) calculation. Data adopted from \([47]\). All parameters are given in W.u.

<table>
<thead>
<tr>
<th>( J_i \rightarrow J_f )</th>
<th>(^{104}\text{Pd} )</th>
<th>(^{106}\text{Pd} )</th>
<th>(^{108}\text{Pd} )</th>
<th>(^{110}\text{Pd} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( B(E2;W_u) )</td>
<td>( B(E2;W_u) )</td>
<td>( B(E2;W_u) )</td>
<td>( B(E2;W_u) )</td>
</tr>
<tr>
<td></td>
<td>exp. sdg-calc.</td>
<td>exp. sdg-calc.</td>
<td>exp. sdg-calc.</td>
<td>exp. sdg-calc.</td>
</tr>
<tr>
<td>(2(_2^+)) ( \rightarrow ) (0(_1^+))</td>
<td>36.9 35.230</td>
<td>34.72</td>
<td>44.3 43.817</td>
<td>42.26</td>
</tr>
<tr>
<td>(0(_2^+)) ( \rightarrow ) (2(_1^+))</td>
<td>13.2 11.673</td>
<td>8.73</td>
<td>43 46.300</td>
<td>56.98</td>
</tr>
<tr>
<td>(2(_2^+)) ( \rightarrow ) (2(_1^+))</td>
<td>21.8 24.880</td>
<td>14.6</td>
<td>39 35.266</td>
<td>29.62</td>
</tr>
<tr>
<td>(2(_2^+)) ( \rightarrow ) (0(_1^+))</td>
<td>1.29 1.283</td>
<td>0.26</td>
<td>1.17 1.170</td>
<td>0.47</td>
</tr>
<tr>
<td>(4(_1^+)) ( \rightarrow ) (2(_2^+))</td>
<td>49</td>
<td>48.63</td>
<td>46.8</td>
<td>71</td>
</tr>
<tr>
<td>(0(_2^+)) ( \rightarrow ) (2(_2^+))</td>
<td>8.375</td>
<td>0.00</td>
<td>0.00</td>
<td>14</td>
</tr>
<tr>
<td>(2(_2^+)) ( \rightarrow ) (0(_2^+))</td>
<td>41.279</td>
<td>0.00</td>
<td>0.00</td>
<td>39</td>
</tr>
<tr>
<td>( \sigma = 1.44 )</td>
<td>( \sigma = 4.18 )</td>
<td>( \sigma = 1.68 )</td>
<td>( \sigma = 5.17 )</td>
<td>( \sigma = 1.29 )</td>
</tr>
</tbody>
</table>
Fig. 3. (Color online.) A comparison between theoretical (sdg-calculation) and experimental $B(E2; (4^+ \rightarrow 2^+))$, $B(E2; (2^+ \rightarrow 0^+))$, $B(E2; (0^+ \rightarrow 2^+))$, $B(E2; (2^+ \rightarrow 0^+))$ and $B(E2; (0^+ \rightarrow 2^+))$ prediction values for Pd Isotopes. The experimental data have been taken from [47].

SO(6) limits in sd IBM and the $U_q(5) \otimes U_g(9)$ and the SO(15) limits in sdg IBM. A sensitive way of testing the signature splitting of the $\gamma$ band is through $S(J)$ given by [68]

$$S(J) = \left( \frac{E(J) - E(J - 1)}{E(J) - E(J - 2)} \frac{J(J + 1) - (J - 1)(J - 2)}{J(J + 1) - J(J - 1)} \right) - 1 \quad (34)$$
Table 5
Comparison of $B(E4)$ from $^{104-110}_{46}$Pd with the sdg calculation. Data adopted from [47]. All parameters are given in W.u.

<table>
<thead>
<tr>
<th>$J_i^\pi \rightarrow J_j^\pi$</th>
<th>$^{104}_{46}$Pd</th>
<th>$^{106}_{46}$Pd</th>
<th>$^{108}_{46}$Pd</th>
<th>$^{110}_{46}$Pd</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>B(E4;W.u)</td>
<td>B(E4;W.u)</td>
<td>B(E4;W.u)</td>
<td>B(E4;W.u)</td>
</tr>
<tr>
<td>exp.</td>
<td>calc.</td>
<td>exp.</td>
<td>calc.</td>
<td>exp.</td>
</tr>
<tr>
<td>$(0)_1^+ \rightarrow (4)_1^+$</td>
<td>0.28</td>
<td>1.76</td>
<td>2.6</td>
<td>2.66</td>
</tr>
<tr>
<td>$(0)_1^+ \rightarrow (4)_2^+$</td>
<td>1.80</td>
<td>1.40</td>
<td>–</td>
<td>1.23</td>
</tr>
<tr>
<td>$(0)_1^+ \rightarrow (4)_3^+$</td>
<td>10.2</td>
<td>8.56</td>
<td>–</td>
<td>7.68</td>
</tr>
<tr>
<td>$(0)_1^+ \rightarrow (4)_4^+$</td>
<td>1.88</td>
<td>7.15</td>
<td>–</td>
<td>1.78</td>
</tr>
<tr>
<td>$(0)_1^+ \rightarrow (4)_5^+$</td>
<td>1.12</td>
<td>7.08</td>
<td>–</td>
<td>4.07</td>
</tr>
<tr>
<td>$\sigma = 0.542$</td>
<td></td>
<td></td>
<td>$\sigma = 0.618$</td>
<td></td>
</tr>
<tr>
<td>$\sigma = 0.67$</td>
<td></td>
<td></td>
<td>$\sigma = 0.71$</td>
<td></td>
</tr>
</tbody>
</table>

Table 6
The coefficients of BE(4) of sdg-IBM used in the calculation of the Pd Isotopes. All parameters are given in $\sqrt{W.u}$.

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>$q_{04}$</th>
<th>$q_{22}$</th>
<th>$q_{24}$</th>
<th>$q_{44}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{104}_{46}$Pd</td>
<td>-1.086</td>
<td>4.0340</td>
<td>-9.903</td>
<td>0</td>
</tr>
<tr>
<td>$^{106}_{46}$Pd</td>
<td>-1.079</td>
<td>4.0273</td>
<td>-10.257</td>
<td>0</td>
</tr>
<tr>
<td>$^{108}_{46}$Pd</td>
<td>-1.031</td>
<td>4.0476</td>
<td>-6.212</td>
<td>0</td>
</tr>
<tr>
<td>$^{110}_{46}$Pd</td>
<td>-1.034</td>
<td>4.0216</td>
<td>-7.837</td>
<td>0</td>
</tr>
</tbody>
</table>

Fig. 4. (Color online.) A comparison between theoretical and experimental $B(E4; (0)_1^+ \rightarrow (4)_1^+)$ prediction values for Pd Isotopes. The experimental data have been taken from [47].

In this section, we present this quantity for some Pd isotopes. In this interpretation the neutron-rich members of these isotopic chains are close to the SO(6) limit of the IBM-1 and thus they fall into the class of nuclei we wish to study in this work. Eq. (34), described gamma-softness...
in sdg-IBM with staggering of \( S(J) \). The results for the \( \gamma \)-band staggering are shown in Fig. 5. In almost all \( \gamma \)-soft nuclei studied, the inclusion of the \( g \)-boson yields a consistently better description of the signature splitting of the \( \gamma \) band. Thanks to this description, we provided a better investigation of triaxiality which can not be explained in the sd boson model without higher boson. Finally, in none of the nuclei studied, we found any evidence for a triaxial minimum.

5.4. Two-neutron separation energies

The shape phase transitions in nuclei can be studied experimentally by considering the behavior of the ground state energies of a series of isotopes, or, more conveniently, the behavior of the two-neutron separation energies, \( S_{2n} \) [27]. On the other hand, the ground-state two-neutron separation energies, \( S_{2n} \), are very sensitive to the details of the nuclear structure. The continuities in the behavior of two-neutron separation energies describe a second-order shape-phase transition between spherical and \( \gamma \)-unstable rotor limits [27,69]. In due course, we have investigated the evolution of two-neutron separation energies along the pd isotopic chains by both experimental and theoretical values, as presented in Fig. 6. The binding energy as a function of proton and neutron number is given by [27]

\[
E_B(N_\pi, N_\nu) = E_c + A_\pi N_\pi + A_\nu N_\nu + \frac{1}{2}B_\pi N_\pi (N_\pi - 1) + \frac{1}{2}B_\nu N_\nu (N_\nu - 1) + C N_\pi N_\nu + E_D(N_\pi, N_\nu)
\]  

Where \( N_\pi (N_\nu) \) is the number of proton/neutron bosons in the valence shell, \( E_c \) is the contribution from the core and \( E_D \) is the contribution to the binding energy due to the deformation. Using Eq. (34), the following relation for the two-neutron separation energy is obtained [27]:

\[
S_{2n}(N_\pi, N_\nu) = E_B(N_\pi, N_\nu) - E_B(N_\pi, N_\nu - 1) = A_\mu + BN_\pi + C_\mu N_\nu + (E_D(N_\pi, N_\nu) - E_D(N_\pi, N_\nu - 1))
\]

Fig. 5. (Color online.) Experimental, sd-IBM and sdg-IBM calculated signature splitting of the \( \gamma \) bands, \( S(J) \), prediction values for Pd Isotopes. The experimental data have been taken from [47].
Fig. 6. (Color online.) A comparison between theoretical and experimental two neutron separation energies, $S_{2n}$ (in keV) for $^{46}_{104}$Pd isotopes. Horizontal line show number of neutron. Experimental data from [47].

Using the $S_{2n}$ empirical values for these isotopic chains [27], we have extracted, $A_n + B_{π} = 21.12$ MeV and $C_n = -0.887$ MeV for pd. Then, we obtained the two-neutron separation energies which are shown in Fig. 6, together with the experimental values. It can be seen from Fig. 6 that exist continuities (linear variation) in the behavior of two-neutron separation energies thus the phase transition for Pd isotopic chains is of second order. Our result confirmed the predictions made in refs. [27,61], suggesting a linear variation of $S_{2n}$ with respect to the neutron number for the transition between spherical and $\gamma$-unstable rotor limits.

5.5. Expectation values of the g-boson number operator

In this section, we present $\langle \hat{n}_g \rangle / N$ quantity for some states of Pd isotopes. In order to obtain the $\langle \hat{n}_g \rangle / N$, for the selected states of the considered nuclei, appropriate quantum number must be used in the Eq. (33). As previously mentioned, shell-model calculation found that $l = 0, 2$ coupled fermion pairs constitute only about 70 percent of the wave functions. Also, according to the discussion done in Ref. [70] the average occupation probability of the g-boson in low-lying levels is less than 10 percent. The calculation under the truncation of the total g-boson number $n_g$ less than two or three gives a sufficiently good approximation for $n$ less than ten.

The results of the calculated $\langle \hat{n}_g \rangle / N$ for the selected states $0_1^+, 2_1^+, 0_2^+, 2_2^+, 4_1^+, 2_3^+, 0_3^+, 3_1^+, 4_2^+, 6_1^+, 2_4^+, 4_3^+, 5_1^+, 6_2^+, 8_1^+, 7_1^+, 8_2^+, 10_1^+, 9_1^+, 10_2^+$ are illustrated in Table 7. We show that the $\langle \hat{n}_g \rangle / N$ can be explained with a simple algebraic method which can calculate all selected states of the considered nuclei. According to the calculations performed for this quantity, we conclude that s and d boson contribute in lower states and g boson contributes in higher states and it could be considered as an effective boson in higher states. We showed that our data for this quantity is almost good and our calculation match very well with other prediction for this quantity.
Table 7
The expectation values of the g-boson number operator for $^{104−110}_{46}$Pd levels.

<table>
<thead>
<tr>
<th>$L^π_J$</th>
<th>$^{104}_{46}$Pd</th>
<th>$^{106}_{46}$Pd</th>
<th>$^{108}_{46}$Pd</th>
<th>$^{110}_{46}$Pd</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0)$^+_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(2)$^+_2$</td>
<td>0.0290</td>
<td>0.0117</td>
<td>0.0191</td>
<td>0.0145</td>
</tr>
<tr>
<td>(0)$^+_2$</td>
<td>0.0246</td>
<td>0.0208</td>
<td>0.0131</td>
<td>0.0134</td>
</tr>
<tr>
<td>(4)$^+_1$</td>
<td>0.0663</td>
<td>0.0745</td>
<td>0.0682</td>
<td>0.0455</td>
</tr>
<tr>
<td>(2)$^+_3$</td>
<td>0.0704</td>
<td>0.0474</td>
<td>0.0567</td>
<td>0.0111</td>
</tr>
<tr>
<td>(0)$^+_3$</td>
<td>0.0327</td>
<td>0.0150</td>
<td>0.0110</td>
<td>0.0137</td>
</tr>
<tr>
<td>(3)$^+_1$</td>
<td>0.1551</td>
<td>0.1159</td>
<td>0.1008</td>
<td>0.0859</td>
</tr>
<tr>
<td>(4)$^+_2$</td>
<td>0.1665</td>
<td>0.1252</td>
<td>0.1079</td>
<td>0.0723</td>
</tr>
<tr>
<td>(6)$^+_1$</td>
<td>0.1022</td>
<td>0.1079</td>
<td>0.0912</td>
<td>0.0619</td>
</tr>
<tr>
<td>(0)$^+_4$</td>
<td>0.0667</td>
<td>0.0527</td>
<td>0.0379</td>
<td>0.0265</td>
</tr>
<tr>
<td>(2)$^+_4$</td>
<td>0.1453</td>
<td>0.1072</td>
<td>0.0763</td>
<td>0.0545</td>
</tr>
<tr>
<td>(4)$^+_3$</td>
<td>0.1140</td>
<td>0.0878</td>
<td>0.0888</td>
<td>0.0777</td>
</tr>
<tr>
<td>(4)$^+_4$</td>
<td>0.1535</td>
<td>0.1156</td>
<td>0.0899</td>
<td>0.0732</td>
</tr>
<tr>
<td>(5)$^+_1$</td>
<td>0.1618</td>
<td>0.1492</td>
<td>0.1224</td>
<td>0.1138</td>
</tr>
<tr>
<td>(6)$^+_2$</td>
<td>0.1618</td>
<td>0.1492</td>
<td>0.1225</td>
<td>0.1143</td>
</tr>
<tr>
<td>(8)$^+_1$</td>
<td>0.1618</td>
<td>0.1492</td>
<td>0.1226</td>
<td>0.1140</td>
</tr>
<tr>
<td>(7)$^+_1$</td>
<td>0.1444</td>
<td>0.1059</td>
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<td>(8)$^+_2$</td>
<td>0.1746</td>
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<tr>
<td>(10)$^+_1$</td>
<td>0.1740</td>
<td>0.1464</td>
<td>0.1263</td>
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<tr>
<td>(9)$^+_1$</td>
<td>0.1743</td>
<td>0.1472</td>
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<tr>
<td>(10)$^+_2$</td>
<td>0.2209</td>
<td>0.2146</td>
<td>0.2103</td>
<td>0.2094</td>
</tr>
</tbody>
</table>

6. Conclusions

In this work, we proposed an algebraic description based on the dual algebraic structure for the three level pairing model to determine the exact energy and eigenstate of the $sdg$ interacting boson model. It provides a simple perspective to understand the complicated $sdg$ Hamiltonian. Our analysis performed via a SU(1, 1)-based Hamiltonian. We have analyzed transition from spherical to $γ$-unstable shapes for $^{104−110}_{46}$Pd. Some observables such as Energy levels, BE(2), BE(4), the two neutron separation energies, signature splitting of the $γ$-vibrational band and expectation values of the $g$-boson number operator are calculated and examined for $^{104−110}_{46}$Pd isotopes. These observables are in agreement with the available experimental data. We have presented an experimental evidence for the spherical to $γ$-unstable shapes transition for $^{104−110}_{46}$Pd. The results indicate that the energy spectra of the $^{104−110}_{46}$Pd isotopes can be reproduced quite well. Our analysis shows that the experimental data are reproduced well enough to satisfactorily describe the key data in $^{104−110}_{46}$Pd. By this description, we provide a better interpretation of some observables such as $BE4$ which cannot be explained in the $sd$ boson model. This model is successful in description of the hexadecapole properties, such as $K = 4^+$ bands and $E4$ tran-
sitions. We specify the BE4 transitions among the lowest 4+ states as an obvious signature for g-boson content in these states. We also present signature splitting of the $\gamma$-vibrational band for some Pd isotopes. In almost all $\gamma$-soft nuclei studied, the inclusion of the g-boson yields a consistently better description of the signature splitting of the $\gamma$-band. Thanks to this description, we provided a better investigation of triaxiality which can not be explained in the sd boson model without higher boson. Finally, in none of the nuclei studied, we found any evidence for a triaxial minimum.

References