Quick and robust meshless analysis of cracked body with coupled generalized hyperbolic thermo-elasticity formulation

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A B S T R A C T
In this article, the MLPG method is applied to the generalized linear coupled thermoelectricity equations. Lord–Shulman modification with a relaxation time parameter is used in the hyperbolic heat conduction equations. A new linear test function which is zero on the boundaries of local test domains is introduced. The test function and its partial derivatives are determined by an exponential RBF approximation method. The approximation of test function and main variables are similar. For the construction of shape functions, neighbors of every point are determined based on the definition of the closest adjacent point pattern. Consequently, test function space becomes independent of trial function space. Direct interpolation method and penalty parameter are used to impose essential boundary conditions. The selection of appropriate parameters are demonstrated in two numerical examples. The small number of used points is the advantage of this method over the FEM that is shown in several examples. The accuracy of results is compared between the meshless method and different analytical and FE solutions. The effect of the relaxation time on SIF under thermal shock is discussed in a separate example. The comparison of meshless results with various examples shows that employed method is accurate and reliable.

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1. Introduction

Although analytical methods provide closed-form solutions for engineering problems, there is much computational and mathematical complexity in such solutions. Thus in order to reach more efficient and general solutions, numerical methods like finite element method [1] or boundary element method [2,3] are required. Until now, the extended finite element method [1,2,4,5] has been widely used as a powerful numerical method of diverse problems like crack growth [5], stationary crack analysis under dynamic or static loads [1,4] and transient analysis of cracked magnetoelectroelastic solids under coupled electro-magneto-mechanical loading [6]. The conventional form of FE methods encounters some difficulties in problems in which remeshing is required, large deformations with high distortions of the mesh exist or crack propagates. Furthermore, the accuracy of the numerical solution depends on the meshing quality. To overcome these problems, the meshless methods appeared since the late of 90s, and now they are developing in different branches of science, especially mechanical engineering. However, the use of these methods may be challenging due to the mathematical complexities and being time-consuming compared with the finite element method. These methods are free of any mesh and only need some scattered points in the configurations [7–10]. In past years, many researchers applied meshless methods to different engineering problems. For example, Tanaka et al. [11,12] used a novel meshfree discretization technique in terms of the reproducing kernel particle method for evaluating mixed-mode SIFs of cracked plates. They employed enriched basis functions for approximation and Voronoi meshing for numerical integration. Nguyen et al. [13] presented extended meshfree Galerkin method based on local partition of unity for modeling of crack growth. They used radial point interpolation method (RPIM) with enriched basis functions for generating the TPS shape functions. Bui et al. [14] used meshfree moving Kriging interpolation method to analyze the natural frequencies of laminated composite plates. Bui et al. [15] numerically analyzed the transient dynamic SIFs of cracked FGMs by extended meshfree methods and extended moving Kriging shape functions. They calculated SIFs by interaction integral method and compared their results with analytical method, XFEM and boundary element ones. Sadamoto et al. [16] studied the buckling of cylindrical shells and calculated the critical buckling loads and their mode shapes by the meshfree reproducing kernel method. Hosseini [17] used a meshless method based on the generalized finite difference method for generalized coupled thermoelectricity analysis based on the Green–Naghdi (GN) theory [18]. Different classification can be considered for meshless methods based on the formulation (global or local), weak form or strong form of the governing equations, type of test functions, approximation method for construction of shape functions and method of enforcing essential boundary

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conditions. The meshless methods which are based on global formulation are not completely meshfree in fact. This means that to calculate the integrals in the weak form a background mesh is required like [11]. Of course, Racz and Bui [19] proposed an adaptive numerical integration method based on mapping techniques for solving domain integrations. Their method maps complex domains to simpler ones applicable to both global and local weak forms. This method is useful but needs some additional calculations of mapping process and a CAD program. Additionally, more calculation time is required. Local meshfree methods such as Petrov-Galerkin method do not need background mesh because of their local nature, nevertheless, the idea of using background mesh is feasible.

In this paper, the weak form meshfree method of local Petrov-Galerkin (MLPG) is applied to the generalized (hyperbolic) coupled linear thermoelectricity governing equations based on Lord–Shulman method with one time lag parameter. Some researchers studied this area by methods rather than meshless method. For example, Alibegloo [20] analytically studied the time-dependent response of sandwich plates with FGM core under thermal shock by using generalized coupled thermoelasticity based on the Lord–Shulman formulation. He used Laplace and Fourier transformations. Heidarpour and Aghdam [21] used differential quadrature method (DQM) to analysis transient response of FGM shells under thermal shock load based on the Lord–Shulman model. Gau and Wang [22] analytically studied the thermal shock fracture of penny-shaped crack based on non-Fourier heat conduction theory by Laplace transform method. Chen and Hu in [23] analytically studied the response of a cracked substrate bonded to a coating using the hyperbolic heat conduction theory and employed Laplace and Fourier transforms. Hosseini et al. used MLPG method in [24] and presented an analytical solution in [25] to study coupled thermoelasticity analysis of FG thick hollow cylinder under thermal shock based on GN theory. Furthermore, thermomechanical and shock loads have been studied in [26,27] by FEM. Liu et al. [28] studied stationary cracks in FG piezoelectric materials (FGPMs) based on the X-FEM under both cooling and heating thermal shocks. They compared FEM results with MLPG ones. Nguyen et al. [27] simulated dynamic and static thermoelastic fracture by extended nodal gradient FE method. In addition, they investigated the simulation of quasi-static crack propagation in complex geometries under thermo-mechanical loading. But, the novelty of this article compared with the mentioned articles is applying MLPG method to the generalized form of the coupled linear thermoelectricity governing equations based on the LS model and considering the second sound effect. The results of numerical investigations of cracked homogenous and FG materials under various time-dependent and time independent thermal and mechanical loads are compared with reference numerical and analytical ones. Thus, the discrete form of the non-Fourier thermoelasticity equations is determined based on the MLPG method and general forms of the mass matrix, damping matrix and stiffness matrix and the force vector are derived. The selection of appropriate parameters of exponential shape function and penalty parameter, numerical integration procedure, the new method of construction of test function based on RBF approximation are explained in details. To evaluate the accuracy of the numerical method, various examples are presented. To calculate the stress intensity factors, the equivalent domain form of J-integral, interaction integral [28–30], are used.

This work is structured as follows. After the introduction, the shape function approximation method and the appropriate parameter selection are explained in the second section. Afterwards, the generation of shape functions and the method of neighbor points allocation is presented in the third section. It is required to enforce discontinuities arising from the crack on the shape functions. This is discussed in the fourth section. Next, in the fifth section, the discretization procedure of the governing equations is presented. Afterwards, the enforcement of the essential boundary conditions and the procedure of numerical integration are discussed in the sixth and seventh sections, respectively. The calculation of the SIFs by interaction integral method and necessary details about it are given in the eighth and ninth sections. Numerical results are presented in the tenth section and they are compared with different analytical and FEM results. In this section, the effect of the relaxation time parameter on the SIFs is investigated under thermal shock loads. Finally, some conclusions are given in the eleventh section.

2. Interpolation/Approximation

Meshfree methods are based on approximating variables (for example, u) in the scattered points by their adjacent points without any mesh. u may be each of displacement components in three directions of an orthogonal coordinate system, temperature or any other engineering variable. For the approximation of such these variables at an arbitrary point x, Eq. (1) is used:

\[ u = \phi \hat{u} \tag{1} \]

In which \( \phi_{ij} \) is the row vector of shape functions and \( \hat{u}_{ij} \) is the unknown column vector of nodal values of the neighbor points. \( n \) is the number of neighbor points within the support domain of point x. The procedure of shape function construction includes different types in meshless methods. Two of the most common methods are the moving least squares and the radial basis functions (RBFS). In this paper, we use the radial basis functions. To get into Eq. (1), first the interpolation is defined as below [31]:

\[ u(x) = \sum_{i=1}^{n} R_i(x)\phi_{ij} + \sum_{j=1}^{m} p_j(x)h_j = R^T(x)a + p^T(x)b \tag{2} \]

\( R \) is the radial basis function vector; \( P \) is the monomials’ vector; \( a \) and \( b \) are unknown vectors. \( m \) is the number of polynomial basis functions. If \( m = 0 \) is selected, it is called classical RBFS and if \( P \) is not empty, it is called the enriched RBF. \( R \) is expressed as below:

\[ R_i(x) = f(r_i) = f\left(\sqrt{(x-x_i)^2 + (y-y_i)^2}\right) \tag{3} \]

This means that each radial function is described as a function of the radial distance between the selected point and its adjacent points. In this paper, exponential RBF is used as described in Table 1 [31]:

In the above relations, \( r_i = \sqrt{(x-x_i)^2 + (y-y_i)^2} \) and \( a \) is the shape parameter. \( A \) is local interpolation area containing adjacent points. In this study, reasonable results have been observed for values of \( a \) between 2 and 3. For all numerical examples, \( a = 2.5 \) has been used. Other RBFS such as TPS, MQ, and CSRBFS are also available for interpolation. For more details, references [31] are recommended.

In two dimensional space, \( P \) is expressed based on the value of \( m \) as follows [32]:

\[ p = \begin{bmatrix} 1 & x & y \end{bmatrix}^T \quad m = 3, \text{ first order} \]

\[ p = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 \end{bmatrix}^T \quad m = 6, \text{ second order} \]

In 2D problems, to stabilize RBFS and approximate the possible linear space appropriately, using first order monomials with \( m = 3 \) has been recommended. Therefore, in this paper \( m = 3 \) is taken.

To find the unknowns \( a \) and \( b \) and to determine \( \phi \), the interpolation has to be applied to all \( n \) points within the support domain of \( x \). In this order, the following relations are obtained [31]:

\[ \hat{u} = R_0 a + p_0 b \tag{4} \]
\[ \dot{\mathbf{u}} = \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \vdots \\ \dot{u}_n \end{bmatrix} \]  
\[ \mathbf{R}_0 = \begin{bmatrix} R_1(r_1) & \cdots & R_1(r_n) \\ \vdots & \ddots & \vdots \\ R_n(r_1) & \cdots & R_n(r_n) \end{bmatrix}_{(n \times n)} \]  
\[ \mathbf{p}_m^T = \begin{bmatrix} 1 & \ldots & 1 \\ x_1 & \ldots & x_n \\ y_1 & \ldots & y_n \\ \vdots & \ddots & \vdots \\ p_m(x_1) & \cdots & p_m(x_n) \end{bmatrix}_{(n \times m)} \]  
\[ \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}^T \]  
\[ \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}^T \]

3. Construction of shape function

In [33], three methods are recommended for the identification of the support domains and construction of the shape functions. Let us consider Gauss method is used for numerical integration. The first way is such that for each Gauss point, a separate action of adjacent points identification and support domain construction is performed. By this method, for Gauss points that are near each other, most of the neighbors will be common. For points far from each other, the number of uncommon points will increase (Fig. 1). Following this method, the support domain of most Gauss points will be different. Accordingly, the construction of shape functions at each Gauss point is performed independently. Thus, due to a large number of Gauss points used in the whole geometry, computation time rises. The second method could be such that if we divide support domains into several subdomains (in order to increase accuracy), every subdomain provides same neighboring points for its relative Gauss points. The third method is that for all Gauss points in all subdomains of the point of interest (point I), adjacent points of point I are used. In this method, the number of support domains will be equal to the total number of points. The speed of the third method is higher than that of the other two methods and the second method has higher speed than the first method. Another different method is that first closer domain point to the intended Gauss point is found and then its neighbor points are used for approximation. In this case, the Gauss point approximation space will be beyond the test function space. For example, we select the domain with bolded boundary in Fig. 1. By this method, approximation in Gauss point g1 is performed using near points of domain point j and for Gauss point g2 this is done by neighboring points of domain point k. Also, it is possible to have main point I among its adjacent points or not. In this paper, we apply this method.

4. Discontinuity treatment

In some problems, we may encounter discontinuities in material properties. For example, in the case of fracture mechanics, displacements along the crack faces are discontinuous. To integrate such discontinuities on RBF approximations, it is necessary to propose a remedy for this problem. Three of available approaches are: visibility method [32,34], diffraction method [32,34], relay method [34–36]. The visibility method is the simplest way for discontinuity treatment in the approximation and construction of shape functions. In this method, first, internal and external boundaries and crack faces are determined. Then, a virtual ray radiating from the selected point to each neighbor point is assumed. If the ray intersects with the boundaries or the cracks faces, that neighbor point is neglected. Fig. 2 shows a general view on the
visibility method. Point O is the considered point and its support
domain is circular. According to the visibility method, only points with
solid circles are taken as neighbors of point O. The hollow circles are
not visible and excluded from the approximation.

In the diffraction method, the virtual point radiating from point O in all
directions refracts at the crack tip. This causes some blind places behind
the crack face to be visible. In this method, taking weight parameter for
new points is inevitable because of the effect of new spotted points for
the construction of shape function is not as much as the effect of the previous
points. For more details, references [32,34–36] are recommended. The
relay method is a completed version of two mentioned methods. The last
two methods are suitable for problems without complex geometries and
strong concaveness on boundaries (Fig. 2b). Details about this method
are reported in [34] comprehensively. In this paper, the first method is
used to describe discontinuities in the construction of shape functions.
For problems of complex geometries, this method may not be efficient
enough.

5. Discrete local weak form of governing equations by MLPG

Transient linear elastic coupled thermoelastic partial differential
equations for the non-homogeneous materials is generally expressed as
follows [37]:

\[
\sigma_{ijkl}(x,t) + b_i = \rho(x)\ddot{u}_k(x,t)
\]

\[
-q_{ij}(x) = \rho(x) c(x) \ddot{\theta}(x,t) + \beta(x) |T_\Omega| \delta_{k,k} \ddot{u}_k(x,t)
\]

(19)

\(\sigma\) is the stress tensor, \(b\) is the body force vector, \(\rho\) is the density, \(u\) is
the displacements vector, \(q\) is the heat flux vector, \(c\) is the specific heat,
\(\theta\) is the temperature difference, \(T_\Omega\) is the initial absolute temperature,
\(x\) is the coordinate vector and \(t\) is the time. \((\ ), j)\) means the partial
derivatives with respect to displacements and \((\ )\) means temporal
derivative. Main variables include the displacements of \(u\) and \(v\) and
the temperature difference of \(\theta\). For displacements, it is necessary to extend
Eq. (16) and the thermal part is expressed according to Eq. (16). This
yields to:

\[
\theta = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_{n-1} \\
\phi_1 & \phi_2 & \cdots & \phi_{n-1} 
\end{bmatrix}
\]

\[
\begin{bmatrix} \tilde{u}_1 \\
\tilde{v}_1 
\end{bmatrix} = \Phi \tilde{\theta}
\]

(20)

\[
\begin{bmatrix} u \\
v 
\end{bmatrix} = \begin{bmatrix} \phi_1 & 0 & \cdots & 0 \\
0 & \phi_1 & \cdots & 0 
\end{bmatrix}
\]

\[
\begin{bmatrix} \tilde{u}_1 \\
\tilde{v}_1 
\end{bmatrix} = \Phi \tilde{\theta}
\]

(21)

In addition to temperature and displacements, there are other variables
including stress, strain and thermal heat flux that have to be defined
based on Eqs. (20) and (21). For an isotropic material, considering
the temperature strains and the thermal heat flux effects, the stress
relation is expressed as follows:

\[
\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{12} \\
\sigma_{22} & \sigma_{22} & \sigma_{22} \\
\sigma_{12} & \sigma_{22} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\
D_{12} & D_{22} & 0 \\
0 & 0 & \frac{\nu}{D_{22}} \end{bmatrix}
\]

\[
\sigma = D\varepsilon - \beta\theta
\]

(22)

\[
f = B\tilde{u} = \begin{bmatrix} \phi_{1,1} & 0 & \phi_{1,2} & \cdots & \phi_{1,n} \\
0 & \phi_{1,2} & \cdots & 0 \\
\phi_{1,2} & \phi_{1,1} & \cdots & \phi_{1,n} \
\phi_{1,2} & \phi_{1,1} & \cdots & \phi_{1,n} 
\end{bmatrix}
\]

(23)

\(B\) is the strain matrix and \(D\) is the material stiffness matrix. Also:

\[
q = \begin{bmatrix} \phi_1 \\
\phi_2 \end{bmatrix} = -k(x) \begin{bmatrix} \phi_{1,1} & \phi_{1,2} & \cdots & \phi_{1,n} \\
\phi_{1,2} & \phi_{1,1} & \cdots & \phi_{1,n} 
\end{bmatrix} = -k(x) \Psi(2\phi(x),\Theta(x))
\]

(24)

\[
\Psi = \begin{bmatrix} \phi_{1,1} \\
\ldots \\
\phi_{1,2} \\
\phi_{1,2} 
\end{bmatrix}
\]

(25)

In which \(k\) is the thermal conductivity coefficient and \(\Psi\) is the thermal
strain matrix.

Now, we can apply the meshless local Petrov–Galerkin method on
the governing equations. In this way, both partial differential equations
are converted from strong form to be weak form using a test function. If
\(R_1\) and \(R_2\) denote, respectively, the first and second PDE, the weak form
of them will be as follows:

\[
\int_{\Omega_i} R_1 d\Omega_i = 0 \quad i = 1, 2
\]

(26)

In which \(v\) is the test function and \(\Omega_i\) denotes the local integration
(support) domain of each domain point. The test function can be
exponential or third/fourth order spline function as recommended in
[32,34]. In this paper, a new linear function is used instead of them that
is presented later.

Eq. (26) is numerically computed at each domain point and local
matrices are assembled to the global matrices. Assuming zero body forces
the first PDE is substituted first:

\[
\int_{\Omega_i} \left( \sigma_{ijkl}(x,t) - \rho(x)\ddot{u}_k(x,t) \right) d\Omega_i = 0 \quad i = 1, 2
\]

Then, using the divergence theorem and some manipulations, the
discrete weak form of the first PDE is obtained:

\[
\int_{\Omega_i} \rho \psi^T \Phi \tilde{\theta} \tilde{u} + \int_{\Gamma_{in+1}} V^T \mathbf{D} \tilde{\theta} d\Gamma - \int_{\Gamma_{in+1}} \nu^T \mathbf{N} D \mathbf{b} d\Gamma \tilde{u} + \int_{\Gamma_{in+1}} \nu^T \beta \Phi \mathbf{d} d\Gamma + \int_{\Gamma_{in+1}} \nu^T \mathbf{N} \phi d\Gamma \tilde{u} = \int_{\Gamma_{in+1}} \nu^T \beta \mathbf{d} d\Gamma
\]

(28)

In which \(\partial\Omega_i = \Gamma_{in+1} \cup \Gamma_L \) and \(\partial\Omega_i\) is the boundary of the local
domain. \(\partial\Omega_i\) consists of three parts. \(\partial\Omega\) is assumed to be the global
boundary. \(\Gamma_{in+1}\) is the part of \(\partial\Omega_i\) that is common with \(\partial\Omega\) and subjected
to tractions, \(\Gamma_w\) is the part of \(\partial\Omega_i\) that is common with \(\partial\Omega\) and the
displacements are defined on it. \(L\) is the part of \(\partial\Omega_i\) without subscription
with \(\partial\Omega\).

\[
\nu^T = \begin{bmatrix} \nu_x \\
-\nu_y 
\end{bmatrix}
\]

(29)

\[
\psi^T = \begin{bmatrix} \psi_x \\
\psi_y 
\end{bmatrix}
\]

(30)

\[
\mathbf{N} = \begin{bmatrix} n_1 \\
n_2 \\
n_3
\end{bmatrix}
\]

(31)
\( \partial \Omega: \text{global boundary} \)

\[ \partial \Omega: \text{local boundary} \]

\[ (\nu = 1) \quad L_\xi (\nu = 0) \quad \Gamma_{st} (\nu \neq 0) \]

\[ (\nu = 0) \]

---

**local test function region**

**Fig. 3.** Different test function definitions.

**neighbor points**

**Gauss point**

**point \( I \) test function region**

---

\[ t' = \{ t_s \quad t_j \}^T \]  
\[ (n_1 \quad n_2)^T \] is normal from \( \partial \Omega \), and \( t' \) is the external load vector applied on the global boundary.

In the case of using a linear test function varying from 1 in the considered point to zero on its local boundary (Fig. 3), the integral on \( L_\xi \) will be equal to zero and the computation’s time is reduced.

When domain point lies exactly on the global boundary \( \partial \Omega \) (Point \( I \)), the new domain is built. Part of the primary domain, presented in dashed line that stays out of the global boundary is neglected. The part of the global boundary that lies inside the local domain along with the internal part of the primary local boundary is taken as the new domain of the test function. Test function and its derivatives will be zero only on the internal boundary as illustrated in Fig. 3. If the whole domain totally lies within the global boundary (Point \( J \)), test function and its derivatives will be zero on the entire local boundary.

Estimation of linear test function and its derivatives is performed by the same procedure used for approximation of main variables as follows (Fig. 4):

\[ v(x) = \{ \phi_1 \quad \phi_2 \quad \ldots \quad \phi_n \} = \phi_I(x) \]

\[ v_s(x) = \{ \phi_{1s} \quad \phi_{2s} \quad \ldots \quad \phi_{ns} \} = \phi_J(x) \]

\[ \phi_I(x) \]

\[ \phi_J(x) \]

---

**So**

\[ v^T = \begin{bmatrix} \phi_1 & 0 & \cdots & 0 \\ 0 & \phi_1 \end{bmatrix} \]  
\[ v_J^T = \begin{bmatrix} \phi_{1s} & 0 & \cdots & 0 \\ 0 & \phi_{1s} \end{bmatrix} \]  
\[ v_T = \begin{bmatrix} \phi_{1t} & 0 & \cdots & 0 \\ 0 & \phi_{1t} \end{bmatrix} \]  
\[ v_J^T = \begin{bmatrix} \phi_{1s} & 0 & \cdots & 0 \\ 0 & \phi_{1s} \end{bmatrix} \]

Using a similar procedure, the discrete equations of the second PDE is expressed as follows:

\[ \begin{bmatrix} \int_{L_\xi} \omega_0 \rho \psi d\Omega \hat{\theta} + \int_{\Omega} \omega_0 \beta T_\xi \Xi d\Omega \hat{\theta} + \int_{\Omega} \omega \rho \psi d\Omega \hat{\theta} \\ \int_{L_\xi} \omega \beta T_\xi \Xi d\Omega \hat{\theta} + \int_{\Omega} kX^T \Psi d\Omega - \int_{\Omega} \omega \kappa \Psi d\Omega + \int_{\Gamma_{sh}} \omega \rho \psi d\Gamma \hat{\theta} \\ - \int_{\Gamma_{sh}} \omega q' \delta \Gamma + \int_{\Gamma_{sh}} \omega h \theta_{\text{in}} d\Gamma \end{bmatrix} \]

\[ = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]

In which \( \omega \) is the test function, \( \theta_{\infty} = T_m - T_0 \) and \( T_m \) is the ambient temperature, \( h \) is the ambient convection coefficient and \( \alpha \) is the thermal expansion coefficient. The temperature difference \( \theta \) is assumed to be negligible compared with the initial temperature \( T_0 \) (\( T_0 \approx T_m \)).

\[ X^T = \begin{bmatrix} (\omega_1 & \omega_2) \end{bmatrix} \]

\[ n = \begin{bmatrix} n_1 & n_2 \end{bmatrix}^T \]

\[ \Xi = \{ \phi_{1,1} \quad \phi_{1,2} \quad \ldots \quad \phi_{n,1} \quad \phi_{n,2} \}^T \]

In Eq. (36), the modified Fourier law of heat conduction (hyperbolic heat conduction theory) with a time lag parameter (i.e. \( t_p \)) based on the Lord–Shulman theory of generalized coupled thermoelasticity has been used. The modified Fourier thermal law is expressed as below [37]:

\[ k_{ij}(x) \theta_j(t,x) + k(x) \theta_{ij}(x,t) = \rho(x) c(x) \left( \theta_i(t,x) + t_p \theta_{i0}(x) \right) + \beta(x) T_0 \left( \theta_{ik}(x,t) + t_p \theta_{ik}(x,t) \right) \]

\[ \beta(x) = a(x) E(x)/(1 - 2\nu(x)) \]

This equation is known as the generalized hyperbolic heat conduction theory according to the Lord–Shulman modification. Applying this modification, the heat conduction equation is transformed from a parabolic form to a hyperbolic one and it guarantees the waveform distribution of temperature (second sound effect) inside the solid body under thermal shock loading or sudden temperature gradients.

The boundary of local domains consists of several sections somehow that \( \partial \Omega_2 = \Gamma_{sh} \cup \Gamma_{sh} \cup \Gamma_{sh} \cup \Gamma_{sh} \). \( \Gamma_{sh} \) is the part of \( \partial \Omega_2 \) that is common with \( \partial \Omega \) and is subjected to the heat flux, \( \Gamma_{sh} \) is the part of \( \partial \Omega_2 \) that is common with \( \partial \Omega \) and the temperature is defined on it, \( \Gamma_{sh} \) is the part of \( \partial \Omega_2 \) with convection and is common with \( \partial \Omega \) and finally, \( L_1 \) is a part of \( \partial \Omega_2 \) that lies within the global boundary.

Thus, the discrete form of governing equations for every domain point is expressed as follows:

\[ M^T \ddot{\theta} + K^T \dot{\theta} + L^T \dot{\theta} = f^T \]

\[ M^{thh} \ddot{\theta} + B^{thh} \dot{\theta} + C^{thh} \theta + L^{thh} \dot{\theta} + \dot{K}^{thh} \ddot{\theta} + K^{thh} \theta = f^{thh} \]
In other words:

\[
\begin{bmatrix}
M^t(1_{2 \times 2N}) & \tilde{u}(1_{2 \times 2}) \\
B^I(1_{2 \times 2N}) & M^I(1_{2 \times 2}) \\
C(1_{2 \times 2}) & S^I(1_{2 \times 2})
\end{bmatrix} + 0(1_{2 \times 2N}) + 0(1_{2 \times 2}) + 0(1_{2 \times 2}) = 0(1_{2 \times 2})
\]

(43)

\[
M^I \tilde{\Delta} + C_I \tilde{\Delta} + K_I \tilde{\Delta} = f_I
\]

(44)

By assembling all local matrices, the general form of discrete governing equations of the coupled linear thermoelastic field considering the second sound effect can be expressed as follows:

\[
\begin{bmatrix}
M(1_{2N \times 2N}) & 0(1_{2N \times 1}) \\
B^I(1_{2N \times 2N}) & M^I(1_{2N \times 2N}) \\
C^I(1_{2N \times 2N}) & S^I(1_{2N \times 2N})
\end{bmatrix} + 0(1_{2N \times 2N}) + 0(1_{2N \times 2N}) + 0(1_{2N \times 2N}) = 0(1_{2N \times 1})
\]

(45)

\[
M_{tot} \tilde{\Delta} + C_{tot} \tilde{\Delta} + K_{tot} \tilde{\Delta} = f_{tot}
\]

(46)

In which \(M_{tot}\) is the global mass matrix, \(C_{tot}\) is the global damping matrix, \(K_{tot}\) is the global stiffness matrix and \(f_{tot}\) is the global force vector.

6. Enforcement of essential boundary conditions

The enforcement of essential or Dirichlet boundary conditions and the selection of appropriate parameters may be challenging in the numerical meshless method. Among different methods, two main methods are used in this paper and both led to reasonable results. The first is the method expressed in [31] that essential boundary conditions are applied directly to the rows and columns of the relevant matrices. For example, if we have a Dirichlet boundary condition for \(j\)th point’s displacement in the \(x\)-direction, e.g. \(u_j = \bar{u}\), first all the relevant rows (the row \(2j - 1\) in matrices \(L, K, M\)) are set to zero. Then, below interpolations are substituted directly into the rows and columns of matrices \(M, K\) and vector \(f\) with global numbering:

\[
\begin{align*}
M(2j - 1, 1 : 2N) &= 0 \\
K(2j - 1, 1 : 2N) &= 0 \\
L(2j - 1, 1 : N) &= 0
\end{align*}
\]

(47)

\[
\phi_1 u_1 + \phi_2 u_2 + \cdots + \phi_n u_n = \bar{u}
\]

(48)

If the \(y\)-direction is constrained, this is imposed on the row \(2j\). Similarly, in the case of temperature constraint, row \(j\) of the related matrices will set to zero. Then, the interpolation is placed directly into the rows and columns of matrices \(L\)th and \(f\th:\)

\[
\phi_1 \theta_1 + \phi_2 \theta_2 + \cdots + \phi_n \theta_n = \bar{\theta}
\]

(49)

The second method is named as penalty method [34]. Despite the previous method, none of the rows of the relevant matrices are changed. Only two new integrals are computed with penalty parameter and then they are added to both sides of the discrete equations as below:

\[
\text{left hand side integrals} + a_{ne} \int_{\Omega_{ne}} \nu \delta \Gamma = \text{right hand side integrals} + a_{ne} \int_{\Gamma_{ne}} \nu' \delta \Gamma
\]

(left hand side integrals) + \(a_{th} \int_{\Omega_{th}} \omega \delta \theta = \) (right hand side integrals) + \(a_{th} \int_{\Gamma_{th}} \alpha \delta \theta\Gamma\)

(50)

Selection of the penalty parameter has a significant effect on the compatibility and accuracy of the computations. Inappropriate values lead to erroneous results, especially in the case of transient analyses. In this paper, the following method is employed to calculate the appropriate penalty parameter:

\[
a_{ne} = 10^p \times \max \left( \left| \text{diag}(M) \right| \right) \quad p = 1 \sim 11
\]

\[
a_{th} = 10^p \times \max \left( \left| \text{diag}(K) \right| \right) \quad p = 1 \sim 11
\]

(51)

7. Numerical integration

One of the complexities of the meshless method and the main difference between this method and finite element method is the numerical calculation of the integrals. The elimination of background mesh leads to some problems if no accurate and efficient alternative method is available. In the meshless methods with the global formulation, it is impossible not to use a partitioning procedure for the numerical integration. Thus, a challenge that may not be irrelevant is to answer the question that are all the meshless methods completely free of mesh or not? The answer may be that if we use geometry partitioning for numerical integration, these methods are not truly meshfree. If integration is free of geometry partitioning, the method will be absolutely meshfree and it is separated completely from the finite element. Of course, we can see the problem from another view that the numerical calculation of all integrals for each point requires specific regions with definite boundaries based on the mathematical nature of the problem. Sum of these specified regions must cover the whole geometry. Thus we never can claim that a meshfree method does not require any geometry partitioning. Necessarily, a procedure should be predicted for partitioning. For example, in [31] the use of simple rectangular or circular regions is mentioned as an
The shape and type of different weight functions have been presented in [49].

Employing the divergence theorem and substituting the strain energy density relation in the transformed equation, equivalent domain form of \( J \)-integral is obtained as follows:

\[
J = \int_{\Gamma_1} \left[ \sigma_{ij} u_{i,1} - \frac{1}{2} \sigma_{ik} \epsilon_{ik} \delta_{ij} \right] q dA + \int_{\Gamma_2} \left[ \sigma_{ij} u_{i,1} - \frac{1}{2} \sigma_{ik} \epsilon_{ik} \delta_{ij} \right] q dA \tag{54}
\]

Using two real and auxiliary fields and applying superposition technique, the above equation can be expressed as

\[
J^{\text{total}} = \int_{\Gamma_1} \left[ \sigma_{ij} u_{i,1}^{aux} + \sigma_{ij} u_{i,1}^{aux} \right] q dA + \int_{\Gamma_2} \left[ \sigma_{ij} u_{i,1}^{aux} + \sigma_{ij} u_{i,1}^{aux} \right] q dA \tag{55}
\]

In which \( J^{\text{total}} \) is concluded from the simultaneous effect of two fields and can be divided into three separate terms:

\[
J^{\text{total}} = J + J^{aux} + II \tag{56}
\]

In this equation, \( II \) represents the interaction integral of the two fields that is obtained by the interaction of the real and auxiliary fields.

\[
II = \int_{\Gamma_1} \left[ \sigma_{ij} u_{i,1}^{aux} + \sigma_{ij} u_{i,1}^{aux} \right] q dA + \int_{\Gamma_2} \left[ \sigma_{ij} u_{i,1}^{aux} + \sigma_{ij} u_{i,1}^{aux} \right] q dA \tag{57}
\]

\[
\text{Eq. (57) is the general form of the interaction integral. In the presence of thermal effects, mechanical strains are used instead of the total strain in the auxiliary field [50].}
\]

\[
II = \int_{\Gamma_1} \left[ \sigma_{ij} u_{i,1}^{aux} + \sigma_{ij} u_{i,1}^{aux} \right] q dA + \int_{\Gamma_2} \left[ \sigma_{ij} u_{i,1}^{aux} + \sigma_{ij} u_{i,1}^{aux} \right] q dA \tag{58}
\]

The auxiliary filed is known a asymptotic crack tip field, \( u^{aux}, \epsilon^{aux} \) and \( \sigma^{aux} \) are functions of the material properties at the crack tip and stress intensity factors \( K_{I}^{aux}, K_{II}^{aux} \). The use of crack tip material properties in the calculation of the auxiliary fields is of importance, especially in the case of heterogeneous material (such as FGM). The asymptotic stress and displacements of the auxiliary field near the crack tip have been reported in [51,52]. For the auxiliary field, different choices can be made. An appropriate selection is Williams’ displacement, stresses and strains of crack tip asymptotic field [53]. They are usable for both homogeneous and non-homogeneous materials. Also, this auxiliary field related to the mechanical loading can also be used in the thermal loading [50].

As presented in [28], the calculation of the SIFs by the interaction integral method for isotropic materials is simpler than that of orthotropic materials. For isotropic materials, by setting \( K_{I}^{aux} = 1 \) and \( K_{II}^{aux} = 0 \) for mode I and \( K_{I}^{aux} = 0 \) and \( K_{II}^{aux} = 1 \) for mode II, coupled modes in the energy release rate equation are simply decoupled and the mode I and mode II SIFs are computed. In this order, using the energy release rate equation in the following general form [54]:

\[
J = (K_{I}^{2} + K_{II}^{2} + (1 + \nu_{tip})K_{I}^{2}K_{II}^{2})/E'_{tip} \tag{59}
\]

and applying superposition technique again, the following equation can be obtained [52]:

\[
J^{\text{total}} = \left( K_{I}^{2} + K_{II}^{2} + (1 + \nu_{tip})K_{I}^{2}K_{II}^{2} \right)/E'_{tip} \tag{60}
\]

In which \( E'_{tip} = E_{tip} \) is related to the plane stress and \( E_{tip} = E_{tip}/(1 - \nu_{tip}^{2}) \) is related to the plane strain condition.

8. Fracture parameters

Rice [48] in 1968 presented the classic form of path-independent \( J \)-integral as follows:

\[
J = \lim_{\Gamma \rightarrow 0} \int_{\Gamma} \left( W \delta_{ij} - \sigma_{ij} u_{i} \right) n_{j} d\Gamma \tag{52}
\]

In which \( W \) is the strain energy density and \( n_{j} \) is the outgoing CCW normal vector of an arbitrary contour \( \Gamma_{0} \) surrounding crack faces (Fig. 6). By applying a weight function \( q \) (such as the plateau weight function) varies from 1 on \( \Gamma_{0} \) to 0 on \( \Gamma_{1} \), the above contour equation is transformed to the following domain form:

\[
J = \lim_{\Gamma_{1} \rightarrow 0} \int_{\Gamma_{1}} \left( \sigma_{ij} u_{i,1} - W \delta_{ij} \right) m_{j} q d\Gamma \tag{53}
\]

In which \( \Gamma = \Gamma_{1} + \Gamma_{+} - \Gamma_{0} + \Gamma_{-} \) is the closed loop limited to two crack faces covering area \( A' \) and enclosing crack tip region.

Fig. 6. Contours and integration domain of interaction integral method.
Assuming two dimensional case and the following appropriate selections, mode I and II stress intensity factors are calculated as below

\[
II = 2(K_I^{aux} + K_{II}^{aux})/E_y^* \\
K_I^{aux} = 1, K_{II}^{aux} = 0 \rightarrow K_I = E_y^*II/2 \\
K_I^{aux} = 0, K_{II}^{aux} = 1 \rightarrow K_{II} = E_y^*II/2
\]

(61)

9. Interaction integral normal aspects

In this paper, details presented in [29] are used for the numerical calculation of the interaction integrals. The main coordinate system used in the numerical calculations is global. The auxiliary field variables and the interaction integral relations include terms of partial derivatives in the local crack tip coordinate system with x-axis parallel to crack faces. It is better to use the global coordinate system for the calculation of all integrals. Therefore, it is necessary to transform the local coordinate system to the global coordinate system for displacement vectors and stress and strain tensors. Thus in this section, the most important points that should be taken into account are presented. The first change in Eq. (58) is such that 1 is replaced by m:

\[
II_m = \int_{\Delta x} \left[ \sigma_{ij}^{aux} u_{m,i} + \sigma_{ij}^{aux} u_{m,j} - \frac{1}{2} (\sigma_{ii}^{aux} + \sigma_{jj}^{aux}) \delta_{ij} \right] q_i dA \\
+ \int_{\Delta x} \left[ \sigma_{ij}^{aux} u_{m,i} + \sigma_{ij}^{aux} u_{m,j} - \frac{1}{2} (\sigma_{ii}^{aux} + \sigma_{jj}^{aux}) \delta_{ij} \right] q_i dA
\]

(62)

The relation between \( H_m \) (global interaction integral vector) and \( H \) (local interaction integral) is expressed as

\[
\begin{align*}
&c = \cos(a) \\
&s = \sin(a)
\end{align*}
\]

(63)

\[
\{ I \}_{\text{local}} = \begin{bmatrix} c & s \end{bmatrix} \{ I \}_{\text{global}}
\]

(64)

In which \( 0 \leq a < 2\pi \) is the crack tip direction angle in the global coordinate system. \( m = 1 \) and \( m = 2 \) represent x and y directions in the global coordinate system respectively. There is no need to transform the stress and strain tensors and the displacements derivatives calculated from the real field. For the auxiliary field variables, expressed in the local crack tip coordinate system, it is necessary to use the following transformations:

\[
\begin{bmatrix} u_{aux}^1 \\ u_{aux}^2 \end{bmatrix}^\text{global} = \begin{bmatrix} c & s \end{bmatrix} \begin{bmatrix} u_{aux}^1 \\ u_{aux}^2 \end{bmatrix}^\text{local}
\]

(65)

\[
\begin{bmatrix} \sigma_{aux}^{11} \\ \sigma_{aux}^{12} \\ \sigma_{aux}^{22} \end{bmatrix}^\text{global} = \begin{bmatrix} c^2 & sc & -2sc \\ sc & c^2 & -2sc \\ -2sc & -2sc & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \sigma_{aux}^{11} \\ \sigma_{aux}^{12} \\ \sigma_{aux}^{22} \end{bmatrix}^\text{local}
\]

(66)

After transforming the tensors and vectors to the global coordinate system, the related derivatives can be calculated using the following relation:

\[
\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}
\]

(68)

So, according to the presented method, the stress intensity factors will be numerically computed.

10. Numerical results

10.1. Static coupled TSIF in a cracked plate with linear and exponential material gradations

Fig. 7 shows a FG plate with an edge crack of length a and exponential material gradation under plane strain condition. The plate is under steady state thermal loading.

This example has been studied in Ref. [50] by the finite element method using 2937 points for two dimensional and 4054 points for three dimensional analysis with enriched elements. In this paper, normalized stress intensity factor is examined using the meshless method and much fewer points, i.e. 236 points including 75 boundary points and 161 internal points are applied.

In the case of exponential gradation, the numerical results are compared against the reference analytical and numerical results in [50,55–59]. The comparison of SIFs between this paper method and references shows that this method is very efficient and leads to desirable results. In this example, a, W and H are assumed to be respectively equal to 0.5, 1 and 8. The exponential variations of Young’s modulus, thermal expansion coefficient and thermal conductivity are defined as below. The

Table 2

<table>
<thead>
<tr>
<th>Case code</th>
<th>( a = 1 )</th>
<th>( a = 2 )</th>
<th>( a = 2.5 )</th>
<th>( a = 3 )</th>
<th>Ref. [50] 2D_1</th>
<th>Ref. [50] 2D_2</th>
<th>Ref. [50] 3D</th>
<th>Ref. [55]</th>
<th>Ref. [59]</th>
<th>Ref. [58]</th>
<th>Ref. [56]</th>
</tr>
</thead>
<tbody>
<tr>
<td>L1G1</td>
<td>0.0136</td>
<td>0.0279</td>
<td>0.0338</td>
<td>0.0407</td>
<td>0.0407</td>
<td>0.0407</td>
<td>0.0406</td>
<td>0.0411</td>
<td>0.0410</td>
<td>0.0409</td>
<td>0.0404</td>
</tr>
<tr>
<td>L1G2</td>
<td>0.0129</td>
<td>0.0129</td>
<td>0.0129</td>
<td>0.0129</td>
<td>0.0129</td>
<td>0.0129</td>
<td>0.0129</td>
<td>0.0129</td>
<td>0.0129</td>
<td>0.0129</td>
<td>0.0129</td>
</tr>
<tr>
<td>L2G1</td>
<td>0.0125</td>
<td>0.0123</td>
<td>0.0123</td>
<td>0.0123</td>
<td>0.0123</td>
<td>0.0123</td>
<td>0.0123</td>
<td>0.0123</td>
<td>0.0123</td>
<td>0.0123</td>
<td>0.0123</td>
</tr>
<tr>
<td>L2G2</td>
<td>0.0338</td>
<td>0.0338</td>
<td>0.0338</td>
<td>0.0338</td>
<td>0.0338</td>
<td>0.0338</td>
<td>0.0338</td>
<td>0.0338</td>
<td>0.0338</td>
<td>0.0338</td>
<td>0.0338</td>
</tr>
</tbody>
</table>
Poisson’s ratio is constant and equals to 0.3 and the initial temperature inside the material is equal to $T_0 = 10$ Celsius.

$$E = E_0 \exp(\beta x) \quad \gamma = \ln \left(\frac{E_2}{E_1}\right)/W$$

Two cases of material properties and four loading conditions are assumed to be

$$L_1 : \begin{cases} E_1 = 1 & E_2 = 5 \\
\alpha_1 = 0.01 & \alpha_2 = 0.02 \\
k_1 = 1 & k_2 = 1 \end{cases} \quad \begin{cases} G_1 : \theta_1 = \theta_2 = 0.5T_0 \\
G_2 : \theta_1 = \theta_2 = 0.05T_0 \end{cases}$$

$$L_2 : \begin{cases} E_1 = 1 & E_2 = 10 \\
\alpha_1 = 0.01 & \alpha_2 = 0.02 \\
k_1 = 1 & k_2 = 10 \end{cases} \quad \begin{cases} G_1 : \theta_1 = 0.2T_0, \theta_2 = 0.5T_0 \\
G_2 : \theta_1 = 0.05T_0, \theta_2 = 0.5T_0 \end{cases}$$

In which $\theta_i$ is the temperature constraint applied to the specified boundaries.

The results of the dimensionless stress intensity factor normalized by $K_0 = E_1 a_1 T_0 \sqrt{\pi a}/(1 - \nu)$ are presented in Table 2. For the first case (L1–G1), despite the equal reduction in the temperature of both sides, the crack is opened due to the presence of non-uniform properties in the FG material.

According to the observed results for the different values of the shape parameter $a$ varying from 1 to 10, the numerical results were closer to the reference ones to the interval of 2–3. In Table 2, the results of the meshless method are presented for four values of the shape parameter. By comparing results between the column related to $a = 2.5$ with other columns, it can be observed that this value for shape parameter may be a better choice, although the value of the optimum parameter can also be a topic of a separate study that is out of the scope of this paper. Little difference between the results of this method and the references finite element or analytical results, demonstrates the good accuracy of the used method. So, for next examples, $a = 2.5$ will be used.

In another example, stress intensity factor for three different crack lengths and two different cases including homogeneous and functionally graded material are examined under the plane strain condition and the following material properties are used:

$$L_1: \frac{a}{W} = 0.3 \quad L_2: \frac{a}{W} = 0.5 \quad L_3: \frac{a}{W} = 0.7$$

**G1: Homogeneous**

$$E = 1.05 \quad \nu = 0.3$$

**G2: FGM**

$$E_1 = 1.05 \quad \alpha_1 = 1.67\nu - 5\nu = 0.3$$

$$E_2 = 5.05 \quad \alpha_2 = 1.00\nu - 5\nu = 0.35$$

In FGM case (G2), the material gradation is assumed to be linear. Thermal loads of $\theta_1 = 0$ and $\theta_2 = 1$ are applied on the selected boundary. Also, The initial temperature ($T_0$) is assumed to be equal zero. Comparison of the results between numerical method and reference [50] shows the accuracy of this method and the proximity of two investigations results (Table 3).

According to these examples and the observed results in the above two tables, it is seen that difference between results is negligible in all cases and this is of high importance compared to the number of points used for this method.

In these examples, we used the following discrete formulation of the governing equation:

$$K_{1\text{tot}} \Delta = f_{\text{tot}} \quad (69)$$

Also, the direct interpolation method used in imposing the essential boundary conditions.

### 10.2. Static SIF in a cracked FG plate under tension and bending loadings

The next numerical example considers a cracked FG plate subjected to two tensile and bending static loading under plane strain condition. In Ref. [30] numerical analysis has been performed using the finite element method with the total number of 2855 points and 927 elements. In this paper, the meshless analysis is performed by 303 points, including 84 boundary points and 219 internal points.

According to the reference, Poisson’s ratio is constant and is assumed to be equal to 0.3 and the variation of Young’s modulus is set to be exponential as below:

$$E_i = E(0) = 1 \quad E_2 = E(1) \quad E_2/E_1 = 0.1, 0.2, 1.0, 5.0, 10.0$$

$$E = E_1 \exp(\beta x) \quad \beta = \ln \left(\frac{E_2}{E_1}\right)/W$$

It is assumed $\sigma_0 = 1$ for tensile loading and $\sigma_0 = 0$ for bending. Now, the results of this method and the references finite element or analytical results, demonstrates the good accuracy of the used method. So, for next examples, $a = 2.5$ will be used.

In another example, which is related to the static mechanical loading, the following discrete form of the governing equations has been used:

$$K \mathbf{U} = \mathbf{f} \quad (70)$$

Furthermore, two methods of enforcing essential boundary condition have shown very close results. So, the direct interpolation method has been used.

### Table 3

Dimensionless SIFs for different values of $a/W$.

<table>
<thead>
<tr>
<th></th>
<th>This paper</th>
<th>Dif. (%)</th>
<th>G2 [60]</th>
<th>This paper</th>
<th>Dif. (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>L1</td>
<td>0.7951</td>
<td>0.7844</td>
<td>1.35</td>
<td>0.4385</td>
<td>0.4294</td>
</tr>
<tr>
<td>L2</td>
<td>0.6705</td>
<td>0.6708</td>
<td>0.04</td>
<td>0.2972</td>
<td>0.2964</td>
</tr>
<tr>
<td>L3</td>
<td>0.4288</td>
<td>0.4344</td>
<td>1.31</td>
<td>0.1444</td>
<td>0.1464</td>
</tr>
</tbody>
</table>

### Table 4

Dimensionless SIFs for different material gradations.

<table>
<thead>
<tr>
<th>$E_2/E_1$</th>
<th>Ref. [30]</th>
<th>This paper</th>
<th>Dif. (%)</th>
<th>Ref. [30]</th>
<th>This paper</th>
<th>Dif. (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>2.544</td>
<td>2.4670</td>
<td>3</td>
<td>1.943</td>
<td>1.8773</td>
<td>3.4</td>
</tr>
<tr>
<td>0.2</td>
<td>2.431</td>
<td>2.3635</td>
<td>2.8</td>
<td>1.706</td>
<td>1.6542</td>
<td>3</td>
</tr>
<tr>
<td>1.0</td>
<td>2.110</td>
<td>2.0644</td>
<td>2.2</td>
<td>1.260</td>
<td>1.2292</td>
<td>0.3</td>
</tr>
<tr>
<td>5.0</td>
<td>1.749</td>
<td>1.7179</td>
<td>1.8</td>
<td>0.924</td>
<td>0.9048</td>
<td>2.3</td>
</tr>
<tr>
<td>10.0</td>
<td>1.588</td>
<td>1.5625</td>
<td>1.6</td>
<td>0.804</td>
<td>0.7884</td>
<td>1.9</td>
</tr>
</tbody>
</table>
Fig. 8. Configuration, domain points (before and after loading), boundary conditions and loadings (example 2).

Fig. 9. Configuration, domain points in different time steps, boundary conditions and loadings.

10.3. **Transient SIF in a cracked homogeneous strip under mechanical shock**

Fig. 9 illustrates a cracked plate under a tensile mechanical shock on its top edge and plane strain condition. The material properties are: Young’s modulus of 210 GPa, Poisson’s ratio of 0.3 and mass density of 8000 kg/m³. The applied tensile stress is equal to 0.5 GPa. This numerical analysis is performed using 411 total points, including 84 boundary points and 327 internal points. In the reference paper [61], the numerical analysis is performed by the extended meshless method in the global formulation and using the enrichment functions at the crack tip.
Fig. 10 shows the dimensionless dynamic stress intensity factor normalized by \( \sigma_0 \sqrt{H/2} \) versus dimensionless time normalized by \( t_c \). The results are in good agreement with the numerical results in [61] and the analytical solution in [62]. The analytical solution (Eq. (71)) is valid until the stress wave reaches the crack tip from the top edge. This time is computed to be equal to \( 3t_c = 3H/2c_d \) according to [61,62] in which \( c_d \) is the dilatational wave speed through the material and is equal to 5946 m/s. Thus, the results are obtained until the approximate time of 0.001 s.

Fig. 9 (b–f) are respectively related to normalized times \( \bar{t} = 0.00, 0.66, 1.44, 1.92, 2.40, 3.00 \). In these figures, the distribution of points and crack opening steps are depicted in several steps. The crack opening has been enlarged by 10 times. It should be noted that from the beginning of loading until when the stress wave reaches the crack tip, the stress intensity factor remains zero and Fig. 10 illustrates this instant exactly.

The analytical solution of stress intensity factor in this example is presented as below:

\[
K_{int}^{dy}(t) = \begin{cases} 
0 & t \leq t_c \\
\frac{2\sigma_t}{1 - v} \left( \frac{c_d(t - t_c)(1 - 2v)}{\pi} \right) & t > t_c
\end{cases}
\]  

(71)

In this example, the following form of the discrete governing equations has been applied:

\[
M\ddot{U} + KU = f
\]  

(72)

According to the observed results, both methods of enforcing essential boundary conditions lead to similar results and the direct method has been applied.

10.4. Transient coupled TSIF in a cracked strip under thermal shock

Fig. 11 illustrates an edged cracked strip subjected to thermal shock loading on its left and right edges under plane strain condition.

In the case of considering functionally graded material, the left hand side is assumed to be made of pure Titanium Carbide (TiC) as the matrix phase and the right hand side is made of Silicon Carbide (SiC) as the inclusion phase with the property values given by Table 5. The variation of the material properties are presented by a micromechanical model as a function of the volume fraction:

\[
V(x) = \left( \frac{x}{W} \right)^\theta
\]

\[
k(x) = k_m \left( 1 + \frac{V(x)(k_i - k_m)}{k_m + (k_i - k_m)(1 - V(x)/3)} \right)
\]

\[
\rho(x) = V(x)\rho_i + (1 - V(x))\rho_m
\]

\[
\alpha(x) = V(x)\alpha_i + (1 - V(x))\alpha_m
\]

Young's modulus and Poisson's ratio of the components are assumed to be constant. Only the thermal properties of conductivity, thermal expansion coefficient and also density are described by a rule of mixture. This type of ceramic/ceramic FGM is utilized in cutting tools and turbines [63]. In the case of when we consider a homogeneous material and not FGM, the homogeneous strip is assumed to be made from Silicon Carbide. So, the mechanical and thermal properties of SiC are applied as the dominant properties of the material.

Due to the symmetry of the configuration and loading, as depicted in Fig. 11, half of the model is used for numerical calculation. In the analytical method, the dimension of the geometry in the vertical direction is assumed to be infinitely large but in the numerical meshless method,
Table 5

Material properties of FGM components (m: matrix phase and i: inclusion phase).

<table>
<thead>
<tr>
<th>Material</th>
<th>E(GPa)</th>
<th>v</th>
<th>α(1e−6/°K)</th>
<th>K(W/mK)</th>
<th>ρ(kg/m³)</th>
<th>c(J/kgK)</th>
</tr>
</thead>
<tbody>
<tr>
<td>m: TiC</td>
<td>400</td>
<td>0.2</td>
<td>7.0</td>
<td>20.0</td>
<td>4900</td>
<td>0.7</td>
</tr>
<tr>
<td>i: SiC</td>
<td>400</td>
<td>0.2</td>
<td>4.0</td>
<td>60.0</td>
<td>3200</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Fig. 12. Comparison of normalized thermal SIFs against normalized time between MLPG and reference numerical results for different values of \( \frac{a}{W} \) (Homogeneous).

Fig. 13. Effect of penalty parameter on thermal SIF for \( \frac{a}{W} = 0.1 \) (Homogeneous).

this dimension is assumed to be 1 unit. Also, the width of 1 unit as per the analytical solution is considered.

The initial temperature is assumed to be \( T_{in} = 0 \). Under the thermal shock loading, the left edge temperature reduces to \( -T_0 \) suddenly (\( T_0 > 0 \)). The right edge temperature is maintained constant and equals to zero. The transient normalized thermal stress intensity factor obtained from the numerical meshless results is compared to the results of \([63]\). In the reference, the analytical uncoupled method with a multi-layer model and Laplace transform has been used to calculate transient TSIFs. So, first the temperature has been calculated separately and then the displacements have been obtained. In this paper, coupled method are used. The difference of results between meshless method and the
analytical solution is observed in Fig. 12. To compare the accuracy of numerical results, the normalized thermal stress intensity factor is plotted versus normalized time for the homogeneous case in Fig. 12. Thermal stress intensity factor and time are normalized by $E_a T_0 \sqrt{\varepsilon W/(1-\nu)}$ and $W^2/k\alpha$ respectively. $\alpha$ denotes thermal diffusivity and is equal to $k/\rho c$. The normalized TSIFs are presented against the normalized time ranging from 0 to 0.3.

According to numerical data, the differences in results are less than 5% for the ratios of $a/W = 0.1, 0.3$. The maximum differences in the numerical and analytical results are related to the maximum normalized stress intensity factor. For larger values of $a/W$, the differences increase because the ratio of crack length to height increases. So, the new dimensions are far from the ideal form of the analytical solution. Thus, the proposed method agrees well with the analytical solution for small cracks and smaller values of $a/W$. It should be noted that in the reference figures, the transient TSIFs were not started from 0, but all the numerical data derived from meshless method are started from 0. In this order, the curves are displaced in the horizontal direction so that we can perform a suitable comparison among different results. Thus, in this example, the numerical TSIFs are not started from 0. This subject also exists in the next comparative curves.

In this example, the following complete discrete form of governing equations has been used as follows:

$$M_{tot} \ddot{\Delta} + C_{tot} \dot{\Delta} + K_{tot} \Delta = f_{tot}$$  \hspace{0.5cm} (73)

According to the observed results, the direct interpolation method did not lead to desired accuracy level. Thus, the second method, i.e. penalty method, with $p=1$ were applied. In Fig. 13 the effect of the penalty parameter on the normalized SIF is depicted. As we see, for $p=0$, the essential boundary condition is not applied accurately and consequently, the maximum normalized SIF is calculated incorrectly. In addition, a delay is seen with respect to the analytical results. For $p=2$, until the maximum TSIF, the curve agrees well with the reference analytical curve but for the times after the maximum TSIF's time, fluctuation and instability are produced in the curve. For $p=1$, the results
are better that can also be observed in Fig. 13. After the values of TSIF attenuates until it reaches steady state, instability is not observed. However, some negligible difference is seen in the maximum TSIF compared against the reference one. Other values of $p$ ranging from 3 to 11, have a behavior similar to that of $p = 2$ that is not depicted in this figure. Thus the selection of $p = 1$ seems to be more reasonable.

In the case of functionally graded material, the values of $q = 1$ and $q = 2$, and the ratios of $a/W = 0.1$ and $a/W = 0.3$ are considered. The normalized TSIFs are compared between the numerical meshless method and the reference solution as depicted in Figs. 14 and 15. Dimensionless time ranges from 0 to 0.1 in these figures.

Now, to investigate the effect of the relaxation time parameter on the TSIFs behavior, a separate investigation is performed. For this, the properties and condition of the previously studied homogeneous material (SiC) are used and the effect of the five relaxation time parameters on the results is examined. The effect of this parameter on SIF is practically truly important under severe thermal shocks and temperature gradients. However, because the intensity of applied thermal shock is not too much, larger time lag parameters are adopted. These parameters are, respectively, 0.1, 0.25, 0.5, 0.75 and 1 s. As it can be observed in Fig. 16a–c, by increasing the time lag parameter, the maximum stress intensity factor rises and the overshooting phenomenon is observed. Furthermore, the pick time is delayed. This pattern can be observed in all three ratios of $a/W$. Also, due to the collision of the second sound waves with the boundaries and diffracting in different directions, an oscillating pattern in the curve of the TSIF can be observed after the peak time. Moreover, a delay is seen in the TSIF rise-up from very first moments of reaching thermal waves to the crack tip in comparison with when the time lag parameter is assumed to be ideally zero. In addition, it seems the maximum overshooting for the small cracks is lesser than that of large cracks. This matter is determined when we observe the maximum dimensionless TSIF for $a/W = 0.5$ doubles in comparison with $a/W = 0.1$. The time delay follows this pattern too. Such that for the larger cracks, the maximum TSIF time delay is more with respect to the smaller crack. Also, it seems that in the steady state, the normalized TSIF slightly decreases compared with the ideal condition.

11. Conclusions

In this paper, meshless local Petrov–Galerkin method is applied to the generalized coupled linear thermoelastic equations with Lord–Shulman modification. Heat conduction and the equation of motion partial differential equations are discretized and consequently, a global second-order and time-dependent linear differential equation with the discrete global matrices of mass, damping and stiffness and force vector resulting from the coupled equations are obtained. To compare the performance of the meshless method, various examples with different thermal and mechanical loads are investigated under steady state and time-dependent (shock load) conditions. Different forms of materials such as homogeneous and heterogeneous (functionally graded) are investigated with linear and non-linear properties profile and the numerical results are compared to the reference analytical and finite element results. The comparison shows the reasonable accuracy in the method.

In this study, we use linear test function. It is approximated using the same RBF used for the approximation of main variables. The approximated test function varies from the unit at the center of the test function domain to zero on its local boundaries. As a result, some integrals are eliminated from the computations. Also, due to the use of similar interpolation procedure, this method is consistent with the approximation of the main variables. The allocation of neighbor points is performed by the pattern of the closest neighbor point determination. The appropriate shape function parameter, under thermomechanical loading and steady state condition, is obtained by matching numerical results with the reference ones and this parameter is used in other examples. According to the results, for values of $a$ ranging from 2 to 3, more favorable results are observed compared to other values. So, 2.5 is selected as the best parameter. The appropriate parameter related to the penalty method ($q$) is observed to be 1. In this order, the variations of thermal stress intensity factors versus time are compared with the reference results for three different values of $q$, i.e., 0, 1 and 2. Results for $q = 0$ are far from the expected values. For $q = 2$ better results are observed for the maximum SIF, but upon approaching the steady state condition, instability is seen in the curve. A similar trend is observed for the large values of $q$. Thus $q = 1$ is selected as an appropriate parameter. Under static and dynamic mechanical shocks, the results of the direct interpolation method are
similar to the results from the penalty method, but under the thermal shock loading, this method is unsuitable because it leads to instability.

After comparing the numerical results with the reference ones, the effect of the Lord–Shulman relaxation time parameter on SIF is investigated in a separate example. According to the results, it is observed that when the relaxation time is increased, maximum SIF’s time is delayed and its value is increased significantly. This pattern is observed for different crack lengths. For larger cracks, the effect of greater relaxation times on the time in which the maximum of normalized stress intensity factor is observed and on the overshooting is impressive than small cracks.

Totally, advantages and disadvantages of this article are listed as below:

Advantages: (a) the small number of used points compared with the finite element method helps to the efficiency of numerical calculations and computation time decreases. (b) Test function used in this method is simple linear function and it is approximated similar to the approximation of main variables by RBFs. Its value varies from 1 at the point of interest to zero on its local internal boundary. This causes some integrals to be zero. Therefore, computational efficiency is enhanced. (c) For every Gauss point, the closest domain point is determined. The adjacent points of the selected domain point are used for approximation at the Gauss point. As a result, this method is compatible with the MLPG method because test function and trial function spaces are independently defined. (d) Generally, enriching functions are used to capture singularities around the asymptotic crack tip field. We do not use them. Due to a large number of basis functions, computation time increases. Instead, the density of points distributed around the crack tip is increased. Also, the density is diminished for regions far from the crack tip. By this way, an equilibrium is established. (e) In this article, a comprehensive discrete form of 2D hyperbolic coupled thermoelasticity governing equations has been presented. Different examples, including mechanical and thermal loads under steady state or transient shock conditions have been examined. It is possible to develop this study to complex 2D configurations, crack growth and 3D problems in the future.

Disadvantages: (a) Only stationary cracks were studied in this article. (b) We presented numerical problems only with simple geometries. Investigation of the effect of the relaxation time on the behavior of crack growth seems to be interesting. Additionally, the comparison between crack growth trajectories under shock loads with and without considering the relaxation time will be an attractive study.

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